Kinetic and Related Models
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EVOLUTIONARY GAME THEORY IN MIXED STRATEGIES: FROM MICROSCOPIC INTERACTIONS TO KINETIC EQUATIONS

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Abstract. In this work we propose a kinetic formulation for evolutionary game theory for zero sum games when the agents use mixed strategies. We start with a simple adaptive rule, where after an encounter each agent increases by a small amount h the probability of playing the successful pure strategy used in the match. We derive the Boltzmann equation which describes the macroscopic effects of this microscopical rule, and we obtain a first order, nonlocal, partial differential equation as the limit when h goes to zero.

We study the relationship between this equation and the well known replicator equations, showing the equivalence between the concepts of Nash equilibria, stationary solutions of the partial differential equation, and the equilibria of the replicator equations. Finally, we relate the long-time behavior of solutions to the partial differential equation and the stability of the replicator equations.

1. Introduction.

Evolutionary game theory, introduced in the 70s by Jonker and Taylor [43] and Maynard Smith [42], is a beautiful mix of biology and game theory. Each player in some population interacts repeatedly with other players by playing a game, obtaining a pay-off as reward or punishment of the pure strategies or actions they used. We can consider now an *evolutive* mechanism, where agents with higher pay-offs have more offsprings, or an *adaptive* one, where less successful players change strategies and imitate the ones which performed better. This process is mathematically formalized using systems of ordinary differential equations or difference equations. Essentially, we have an equation describing the evolution of the proportion of players in each pure strategy, usually in the form of a rate equation where the growth rate is proportional to the *fitness* of the strategy. The so-called replicator equations (see (1.2) below) is a famous and well-studied example of such systems. However, observe that the fitness also evolves, since it depends on the distribution of the population on the strategies, so interesting problems appear concerning the existence and stability of fixed points, and their relationship with

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the Nash equilibria of the underlying game. We refer the interested reader to the monographs [15, 23, 39] for details.

Similar mechanisms when players use mixed strategies are less frequent in the literature. Recently, in [37] we introduced an evolutive model for finitely many players, and we obtained a systems of ordinary differential equations describing the evolution of the mixed strategy of each agent. The number of equations is then proportional to the number of agents. Thus, the limit of infinitely many agents seems to be intractable in this way. However, a simple hydrodynamic interpretation is possible, leading to a first order partial differential equation modeling the strategy updates: we can think of the players as immersed in a fluid, flowing in the simplex Δ of mixed strategies, following the drift induced by the gradient of the fitness of the strategies given the distribution of the whole population on this simplex.

Let us present a brief outline of our model and the main results in this work, and see Section §2 for the precise definitions, notations, and previous results. We consider a population of agents playing a finite, symmetric, zero sum game, with pay-off matrix A. Each player starts with a given mixed strategy, i.e., a probability distribution on the set of pure strategies. They are randomly matched in pairs, and select at random a pure strategy using their respective mixed strategies which they use to play the game. After the match, each player changes its mixed strategy by adding a small quantity h to the winner strategy, and reducing the loosing one in the same amount. Some care is necessary in the definition of h to avoid values greater than one, or less than zero. This is needed only near the boundary of the simplex so we replace the constant h by a function $h(p)$ satisfying $h(p) < dist(p, \partial\Delta)$. This adaptive, microscopic rule, first introduced and analyzed in [37] for finitely many players, induces a flow of the players in the simplex, whose study is the main purpose of this paper.

Let us call u_t^h the distribution of agents on the simplex. We can think of $u_t^h(p)dp$ as the probability to find a player with a strategy q in a cube of area dp centered at p. Now, it is possible to describe the time evolution of u_t^h with a Boltzmann-like equation, whose collision part reflects the dynamics in the changes of strategies due to encounters. This procedure is strongly inspired by the kinetic theory of rarefied gases and granular flows and has been successfully implemented to model a wide variety of phenomena in applied sciences (see e.g. [5, 6, 7, 33, 34, 35, 36] and the surveys in Ref. [29] for further details).

However, Boltzmann-like equations are challenging objects to study. Performing the so-called grazing limit or quasi-invariant limit (see [16, 18, 17, 33]), we can approximate it by a Fokker-Planck equation which is satisfied by $v_t = \lim_{h\to 0} u_{t/h}^h$. Besides the intrinsic stochastic nature of the interactions, we can add a small noise in the agents' movements. The Fokker-Planck equation then reads

$$\frac{\partial v_t}{\partial t} + div(\mathcal{F}[v_t]v_t) = \lambda \sum_{i,j=1}^{d} Q_{i,j} \frac{\partial^2}{\partial p_i \partial p_j}(Gv_t),$$

where Q and G depend on v_t and the intensity of the noise, $\lambda \geq 0$ depends on the ratio of the noise to the convection term, and the vector-field $\mathcal{F}[v_t]$, which depends on v_t, is given by

$$\mathcal{F}[v_t] = h(p)[p_i e_i^T A\bar{p}(t) + \bar{p}_i(t)e_i^T Ap],$$

with $\bar{p}(t) = \int_\Delta p\, dv_t(p)$ the mean strategy at time t.

In particular, when $\lambda = 0$, i.e., when the convection term dominates the noise, we obtain the first order, nonlocal, mean field equation

$$\frac{\partial v}{\partial t} + div(\mathcal{F}[v_t]v_t) = 0. \tag{1.1}$$

Existence and uniqueness of solutions for (1.1) follows by using the classical ideas of [9, 19, 30] (see also [11, 22]).

One of the main focus of the paper is the study of the long-time behavior of solutions to (1.1) and the stability of the stationary solutions of the form $v = \delta_p$, where δ_p is a Dirac mass at p, which corresponds to the case where all the players use the same mixed strategy p.

These issues are closely related to the behaviour of the integral curves of the vector-field $\mathcal{F}$. It is worth noticing the close resemblance of $\mathcal{F}$ with the replicator equations

$$\frac{dp_i}{dt} = p_i(e_i^T Ap - p^T Ap). \tag{1.2}$$

We can thus expect a relationship between the long-time behaviour of the solutions of (1.1) and of the solutions to the replicator equations (1.2). A well-known result in evolutionary game theory, known as the Folk Theorem (see [24] and Theorem 2.1 below), relates the long-time behaviour of the solution of the replicator equation to the Nash equilibria of the zero-sum game with pay-off matrix A.

Our first main theorem can be thought of as a generalization of the Folk Theorem. Indeed we prove that the following statements are equivalent:

- p is a Nash equilibrium of the game.
- δ_p is a stationary solution to (1.1),
- p is an equilibrium of the replicator equations (1.2),
- $Ap = 0$, where A is the pay-off matrix of the game.

Our second main theorem states that if v_t is a solution to (1.1), then the mean strategy of the population,

$$\bar{p} = \int_\Delta p\, dv_t$$

is a solution to the replicator equations (1.2) while it stays in $\{h = c\}$. See Section §5 for the precise statement of both theorems.

Finally, we show some results about the asymptotic behaviour of the solution v_t of (4.6) and their relationship with the game with pay-off matrix A.

In the simplest case, namely the case of a two-strategies game, we can precisely describe the asymptotic behavior of v_t. Then, we turn our attention to symmetric games with an arbitrary number of strategies. Following [39], a zero-sum symmetric game has no stable interior equilibria, and periodic solutions to the replicator equations appear as in the classical Rock-Paper-Scissor. So, we will show that if all the trajectories of the replicator equations are periodic orbits, then v_t is also periodic.

Let us compare briefly other works dealing with similar issues. To our knowledge, the first work dealing with evolutionary game theory for mixed strategies is due to Boccabella, Natalini and Pareschi [8]. They considered an integro-differential equation modeling the evolution of a population on the simplex of mixed strategies following the idea behind the replicator equations. That is, the population in a fixed strategy p will increase (respectively, decrease) if the expected pay-off given the distribution of agents in the simplex is greater than (resp., lower than) the mean pay-off of the population. A full analysis of the stability and convergence to equilibria is performed for binary games. Let us remark that their dynamics inherit several properties of the replicator equations, and if some mixed strategy p has zero mass in the initial datum, the solution will remain equal to zero forever. So, agents cannot learn the optimal way of play, and they are faced to extinction or not depending on the mixed strategies which are present in the initial datum. In some sense, this is equivalent to consider each mixed strategy as a pure one in a new game. The mathematical theory for infinitely many pure strategies was developed by Oeschssler and Riedel in [31], and studied later by Cressman [14], Ackleh and Cleveland [13], and also Mendoza-Palacios and Hernández-Lerma [27].

Let us mention also [4] which deals with an evolutionary model on the set of mixed strategies using the replicator dynamics. As in [8], the authors propose an evolutionary update based on the birth/death of agents which is proportional to their pay-offs compared with the mean population pay-off. It follows that agents located at less successful strategies tend to vanish while the other ones increases, thus yielding a Darwinian dynamic. In contrast in our model, agents are learning since they evolve in the simplex, searching for successful strategies (of course, successful relative to the population distribution), and the flow of agents in the simplex is governed by a flux which, surprisingly, turns out to be related to the replicator system for finitely many pure strategies. In particular in our model the replicator equation emerged from the interaction rule instead of being imposed from the start.

Finally, we can cite also [2, 26, 38, 45] where binary (or more general) games have pay-offs depending on some real parameters, and they study the distribution of players on this space of parameters (say, wealth, velocity, opinion, among many other characteristics). Now, the strategy that they play in the binary game is selected using partial or total knowledge of the global distribution of agents. For example, we can assume that x represent the wealth of an agent, and in the game they will cooperate or not depending on the median of the society; if they cooperate, a fraction of wealth is transferred from the richer to the poorer, on the other hand, if they does not cooperate, no changes occur.

The paper is organized as follows. In Section §2 we first recall some basic results concerning game theory, the replicator equations and measure theory to make the paper self-contained. We then present in Section §3 the model we are concerned with, and we deduce in Section §4 the partial differential equations allowing to study the long-time behaviour of the system. In Section §5 we prove the generalization of the Folk Theorem. Section §6 is devoted to the study of the asymptotic behaviour and the stability of the stationary solutions to the mean-field equation (1.1) and their relationship with the replicator equations, and we present agent based and numerical solutions of the model. The lengthy or technical proofs of existence and uniqueness of solutions of the relevant equations are postponed to the Appendix for a better readability of the paper.

2. **Preliminaries.**

2.1. **Preliminaries on game theory.** We briefly recall some basics about game theory and refer to the excellent references available on general game theory for more details (e. g. [25]). Since we are concerned in this paper with two players, symmetric, zero-sum, finite games in normal form, we will limit our exposition to this setting.

A two-players finite game in normal form consists of two players named I and II, two finite sets $S^1 = \{s_1, \dots, s_{d_1}\}$ and $S^2 = \{\tilde{s}_1, \dots, \tilde{s}_{d_2}\}$, and two matrices

$A = (a_{ij})_{1\le i\le d_1, 1\le j\le d_2}$, $B = (b_{ij})_{1\le i\le d_1, 1\le j\le d_2} \in \mathbb{R}^{d_1\times d_2}$. The elements of S^1 (respectively, S^2) are the pure strategies of the first (resp., second) player and model the actions it can choose. Once both players have chosen a pure strategy each, say I chose $s_i \in S^1$ and II chose $\tilde{s}_j \in S^2$, then I receives the pay-off $u^1(s_i, \tilde{s}_j) := a_{ij}$ and II receives $u^2(s_i, \tilde{s}_j) := b_{ij}$. Thus the pay-off received by each player depends on the pure strategies chosen by each players and on their pay-off functions u^1 and u^2. It is convenient to identify s_i with the the i-th canonical vector e_i of $\mathbb{R}^{d_1}$, and $\tilde{s}_j$ with the the j-th canonical vector e_j of $\mathbb{R}^{d_2}$. This way the pay-off of I and II are

$$u^1(s_i, \tilde{s}_j) = e_i^T A e_j = a_{ij}, \qquad u^2(s_i, \tilde{s}_j) = e_i^T B e_j = b_{ij}.$$

The game is said:

- symmetric, when I and II are indistinguishable both from the point of view of the sets of actions available and the pay-off functions:
  $S^1 = S^2 =: S$, and
  $u^1(s, \tilde{s}) = u^2(\tilde{s}, s)$ for any $(s, \tilde{s}) \in S \times S$.
  This last equality means that $A = B^T$.
- zero-sum if $u^1 + u^2 = 0$, i.e. $B = -A$. This means that the gain of one player is exactly the loss of the other one.

Notice in particular that the game is symmetric and zero-sum if and only if $A^T = -A$, in other words, the matrix A is antisymmetric.

We illustrate the above definitions with the popular Rock-Paper-Scissors game. This is a two-players zero-sum game with pure strategies

$$S^1 = S^2 = \{Rock, Paper, Scissors\} = \{e_1, e_2, e_3\}$$

(we identifed as before each pure strategy with the canonical vectors of $\mathbb{R}^3$). We then define the pay-off matrix A of the first player as

$$A = \begin{pmatrix} 0 & -a & b \\ b & 0 & -a \\ -a & b & 0 \end{pmatrix} \tag{2.1}$$

where a, $b > 0$, and the pay-off matrix of the second player is $B = -A$ being the game zero-sum by definition. For instance if I plays Paper and II plays Rock then I earns $a_{21} = b$ (so II looses b), and if I plays Scissors and II Rock then I earns $a_{32} = -a$ (and thus II gains a). When $a = b$ then A is antisymmetric meaning that the game is symmetric.

A central concept in game theory is that of Nash equilibrium. A Nash equilibrium is a pair $(s^*, \tilde{s}^*) \in S^1 \times S^2$ such that neither player has incentive to change its action given the action of the other player: there hold at the same time

$$u^1(s^*, \tilde{s}^*) \ge u^1(s, \tilde{s}^*) \qquad \text{for any } s \in S^1,\ s \ne s^*,$$

and

$$u^2(s^*, \tilde{s}^*) \ge u^2(s^*, \tilde{s}) \qquad \text{for any } \tilde{s} \in S^2,\ \tilde{s} \ne \tilde{s}^*.$$

A Nash equilibrium thus models a status quo situation. When the above inequalities are strict, the Nash equilibrim is said to be strict. However, there not always exists a Nash equilibrium in pure strategies as can be seen for instance in the Rock-Paper-Scissors game (2.1): by symmetry, if both players are playing some fixed pure strategies, at least one has incentive to change to another strategy.

The main mathematical issue here is the lack of convexity of the strategy spaces S^1 and S^2. This motivates the introduction of mixed strategies as probability measures over the set of pure strategies: a mixed strategy for I is a vector $p =$

$(p_1, \dots, p_{d_1})$ where p_i is the probability to play the i-th pure strategy s_i. Thus $p_i \in [0, 1]$ and $\sum_i p_i = 1$. Remember that we identify the pure strategies with the canonical vectors of $\mathbb{R}^{d_1}$. Let

$$\Delta_1 = \{p = (p_1, \dots, p_{d_1}) \in \mathbb{R}^{d_1} : p_1, \dots, p_{d_1} \geq 0, \sum_i p_i = 1\}$$

be the simplex in $\mathbb{R}^{d_1}$, and similarly we denote Δ_2 the simplex in $\mathbb{R}^{d_2}$. We extend the pay-off functions u^1 and u^2 to $\Delta_1 \times \Delta_2$ as expected pay-offs in the following way: for $(p, \tilde{p}) \in \Delta_1 \times \Delta_2$,

$$u^1(p, \tilde{p}) = \sum_{i,j} p_i \tilde{p}_j u^1(s_i, \tilde{s}_j) = \sum_{ij} p_i \tilde{p}_j a_{ij} = p^T A \tilde{p}$$

and similarly,

$$u^2(p, \tilde{p}) = p^T B \tilde{p}.$$

We can then extend the notion of Nash equilibrium to mixed strategies saying that $(p^*, \tilde{p}^*) \in \Delta_1 \times \Delta_2$ is a Nash equilibrium if at the same time

$$u^1(p^*, \tilde{p}^*) \geq u^1(p, \tilde{p}^*) \qquad \text{for any } p \in \Delta_1,\ p \neq p^*,$$

and

$$u^2(p^*, \tilde{p}^*) \geq u^2(p^*, \tilde{p}) \qquad \text{for any } \tilde{p} \in \Delta_2,\ \tilde{p} \neq \tilde{p}^*.$$

Nash's celebrated Theorem states that a finite game in normal form always has a Nash equilibrium in mixed strategies, we refer to [21] for a simple proof. Moreover, when the game is symmetric (so that $S := S^1 = S^2$) there always exists a Nash equilibrium of the form $(p^*, p^*) \in \Delta \times \Delta$. For instance, in the symmetric Rock-Paper-Scissors game (2.1) with $a = b$, the unique symmetric Nash equilibrium is $p^* = (1/3, 1/3, 1/3)$. This means that no players has an incentive to deviate when they choose their action with equiprobability. Of course, for zero sum games, the existence of an equilibria goes back to Von Neumann's Minimax Theorem.

2.2. **The replicator equations.** The concept of Nash equilibrium is quite static and computationally challenging. Various dynamic have been proposed to model the learning process of a Nash equilibrium by an individual. A popular one is a system of ordinary differential equations known as replicator equation that was introduced by Taylor and Jonker in [43] (see also [40, 42]).

Consider a large population of individuals randomly matched in pairs to play a two-player symmetric game with pay-off matrix $A \in \mathbb{R}^{d \times d}$. The players are divided into d groups according to the pure strategy they use. Let $p(t) = (p_1(t), \dots, p_d(t))$, where $p_i(t)$ is the proportion of individuals playing the i-th pure strategy e_i at time t. We want to write down an equation for $p_i'(t)$, $i = 1, \dots, d$, modelling the fact that individuals playing a strategy with high fitness should be favored and they will produce more offsprings than individuals playing a low fitness strategy.

In the case of the replicator equation, the fitness of an individual playing the i-th strategy is defined as the difference between the expected pay-off received against a randomly selected individual, and the expected pay-off received by a randomly selected individual playing against another randomly selected individual. Thus, the fitness of the i-th strategy e_i is $e_i^T A p(t) - p(t)^T A p(t)$. Notice that the fitness depends on the distribution of strategies in the whole population and changes in time. We then assume that agents with positive (respectively, negative) fitness have a reproductive advantage (resp., disadvantage) leading to their reproduction (resp., death) at a rate proportional to their fitness. We thus arrive at the replicator equations,

$$\frac{d}{dt}p_i = p_i(e_i^T Ap - p^T Ap) \qquad i = 1, \dots, d. \tag{2.2}$$

It is easily shown that if $p(0) \in \Delta$, where Δ is the simplex of $\mathbb{R}^d$, then $p(t) \in \Delta$ for all time $t \geq 0$.

There is a strong connection between the rest point of the system (2.2) and the Nash equilibria of the game with pay-off matrix A as stated in the so-called Folk Theorem:

Theorem 2.1. *Let us consider a two player symmetric game in normal form with finitely many pure strategies. Then*

1. *if $(p, p) \in \Delta \times \Delta$ is a symmetric Nash equilibrium, then p is a rest point of* (2.2)*,*
2. *if (p, p) is a strict Nash equilibrium, then it is an asymptotically stable rest point of* (2.2)*,*
3. *if the rest point p is the limit as $t \to +\infty$ of a trajectory lying in the interior of Δ then (p, p) is a Nash equilibrium,*
4. *if the rest point p is stable then (p, p) is a Nash equilibrium.*

Moreover, none of the converse statements holds.

We refer to the surveys [24, 28]) for a proof and related results on the replicator equations.

2.3. **Preliminaries on probability measures and transport equations.** We denote by $\mathcal{M}(\Delta)$ the space of bounded Borel measures on Δ and by $\mathcal{P}(\Delta)$ the convex cone of probability measures on Δ. We denote by $\|.\|_{TV}$ the total variation norm on $\mathcal{M}(\Delta)$ defined as

$$\|\mu\|_{TV} = \sup_{\|\varphi\|_\infty \leq 1} \int_\Delta \varphi \, d\mu.$$

However, the total variation norm will be too strong for our purpose and it will be more convenient to work with the weak*-convergence. We say that a sequence $(\mu_n)_n \subset \mathcal{P}(\Delta)$ converges weak* to $\mu \in \mathcal{P}(\Delta)$ if

$$\int_\Delta \varphi \, d\mu_n \to \int_\Delta \varphi \, d\mu \qquad \text{for any } \varphi \in C(\Delta).$$

It is well-known that, since Δ is compact, $\mathcal{P}(\Delta)$ is compact for the weak*-topology. The weak*-topology can be metricized in many ways. It will be convenient to consider the Monge-Kantorovich or Wasserstein distance on $\mathcal{P}(\Delta)$ defined as

$$W_1(u, v) := \sup \left(\int_\Delta \varphi(p) \, du(p) - \int_\Delta \varphi(p) \, dv(p) \right),$$

where the supremum is taken over all the Lipschitz functions φ with Lipschitz constant $Lip(\varphi) \leq 1$. It is known that W_1 is indeed a distance that metricizes the weak*-topology, see [46].

We will work in this paper with first order partial differential equations of the form

$$\partial_t \mu_t + \operatorname{div}(v(x)\mu_t) = 0 \qquad \text{in } \mathbb{R}^d \tag{2.3}$$

with a given initial condition $\mu_0 \in P(\mathbb{R}^d)$ and where $v : \mathbb{R}^d \to \mathbb{R}^d$ is a given vector-field usually assumed to be bounded and globally Lipschitz. A weak solution to this equation is $\mu \in C([0, +\infty), P(\mathbb{R}^d))$ satisfying

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t = \int_{\mathbb{R}^d} \varphi \, d\mu_0 + \int_0^t \int_{\mathbb{R}^d} v(x) \cdot \nabla\varphi(x) \, d\mu_s(x) ds \qquad \text{for any } \varphi \in C_c^\infty(\mathbb{R}^d).$$

Let $T_t : \mathbb{R}^d \to \mathbb{R}^d$ be the flow of v defined for any $x \in \mathbb{R}^d$ by

$$\begin{aligned} \frac{d}{dt} T_t(x) &= v(T_t(x)) \qquad \text{for any } t \in \mathbb{R}, \\ T_{t=0}(x) &= x. \end{aligned}$$

It is well-known (see e.g. [46]) that equation (2.3) has a unique weak solution given by $\mu_t = T_t \sharp \mu_0$, the push-forward of μ_0 by T_t. This means that $\int_{\mathbb{R}^d} \varphi \, d\mu_t = \int_{\mathbb{R}^d} \varphi(T_t(x)) \, d\mu_0(x)$ for any φ bounded and measurable. This result, which is simply a restatement of the standard characteristic method, can be generalized to deal with equations with a non-autonomous vector-field $v(t, x)$ assumed to be continuous and globally Lipschitz in x, uniformly in t.

3. **Description of the model.** We consider a finite population of agents. Two randomly selected agents meet, they play a game, and then update their strategies taking into account the outcome of the game. The game played during an interaction is always the same. It is a two-player, zero-sum game with a set $\{s_1, \ldots, s_d\}$ of pure strategies and whose pay-off is given by a matrix $A = (a_{lm})_{1 \le l,m \le d} \in \mathbb{R}^{d \times d}$. We will assume the game is symmetric, i.e., $A^T = -A$, and with out loss of generality we take $|a_{lm}| \le 1$ for any $l, m = 1, \ldots, d$.

Each agent i has a mixed strategy $p = (p_1, \ldots, p_d) \in \Delta$. Here p_l is the probability that agent i plays the l-th pure strategy s_l.

When two agents i and j meet and play the game, they update their respective strategies using a myopic rule, both agents increase by $\delta h(p) > 0$ the probability of playing the winning strategy and decrease by $\delta h(p) > 0$ the loosing one. Here, δ is a small positive parameter, and $h(p)$ is a positive function of p, to ensure that the updated strategy p^* remains in Δ. For instance, we can take

$$h(p) := \min \left\{ \prod_{i=1}^d p_i, c \right\}. \tag{3.1}$$

where $c < 1$ is a positive constant, and hence $h(p) \to 0$ as $p \to \partial\Delta$.

More precisely, if the pure strategies s_l and s_m were played, agent i only updates the probabilities p_l, p_m to p_l^*, p_m^* as follows

$$\begin{aligned} p_l^* &= p_l + a_{lm} \delta h(p), \\ p_m^* &= p_m - a_{lm} \delta h(p). \end{aligned} \tag{3.2}$$

Agent j updates the probabilities $\tilde{p}_l$, $\tilde{p}_m$ in the same way. Notice that probabilities are raised/lowered proportionally to the gain/loss $a_{lm}\delta$.

To model the choice made by agent i of which pure strategy to play, we fix a random variable ζ uniformly distributed in $[0, 1]$ and then consider the random vector $f(\zeta; p) = (f_1(\zeta; p), \ldots, f_d(\zeta; p))$ where

$$f_i(\zeta; p) := \begin{cases} 1 & \text{if } \sum_{j<i} p_j \le \zeta < \sum_{j \le i} p_j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Notice that $f_i(\zeta;p) = 1$ with probability p_i. Agent j fixes in the same way a random variable $\tilde{\zeta}$ uniformly distributed in $[0,1]$. Then $f(\zeta,p)^T A f(\tilde{\zeta},\tilde{p}) \in [-1,1]$ is the payoff of i when playing against j (recall that the coefficient of A belongs to $[-1,1]$). We can thus rewrite the updating rule (3.2) as

$$p_i^* = \begin{cases} p_i + \delta h(p) f(\zeta,p)^T A f(\tilde{\zeta},\tilde{p}) & \text{if } f_i(\zeta,p) = 1 \text{ and } f_i(\tilde{\zeta},\tilde{p}) = 0, \\ p_i - \delta h(p) f(\zeta,p)^T A f(\tilde{\zeta},\tilde{p}) & \text{if } f_i(\zeta,p) = 0 \text{ and } f_i(\tilde{\zeta},\tilde{p}) = 1, \\ p_i & \text{otherwise.} \end{cases}$$

We can also add a small noise to p_i^* in the following way. We fix $r > 0$ small enough so that $(\delta + r) < 1$, and a smooth function G such that $G(p) \leq p_i$ for any $p \in \Delta$ and any i like e.g. $G(p) = h(p)$. We then consider a random vector q uniformly distributed over Δ (we refer to [32] and [47] for some possible algorithms to generate uniform random vectors on the simplex). The additive random noise is then taken as $r(q_i - 1/d)G(p)$.

We thus arrive at the following interaction rule:

Definition 3.1. Consider an agent with strategy p interacting with an agent with strategy $\tilde{p}$ through the game defined by the matrix A. They update their strategies from p to p^*, and $\tilde{p}$ to $\tilde{p}^*$, as follows

$$\begin{aligned} p^* &= p + \delta h(p) f(\zeta,p)^T A f(\tilde{\zeta},\tilde{p}) \Big(f(\zeta,p) - f(\tilde{\zeta},\tilde{p}) \Big) + r(q - \vec{1}/d) G(p), \\ \tilde{p}^* &= \tilde{p} + \delta h(\tilde{p}) f(\zeta,p)^T A f(\tilde{\zeta},\tilde{p}) \Big(f(\zeta,p) - f(\tilde{\zeta},\tilde{p}) \Big) + r(\tilde{q} - \vec{1}/d) G(\tilde{p}), \end{aligned} \tag{3.4}$$

where $\vec{1} = (1,\ldots,1) \in \mathbb{R}^d$.

Let us remark that p^* and $\tilde{p}^*$ are random variables. Indeed there are two sources of randomness in the updating rule. First, there is the presence of the random variables ζ and $\tilde{\zeta}$ which model the fact that the players choose the pure strategy they are about to play at random using their mixed strategy $p, \tilde{p}$. The second factor of randomness is the noise $r(q - \vec{1}/d)G(p)$.

Let us verify now that p^* remains in the simplex Δ.

Lemma 3.1. *The strategy p^* belongs to Δ.*

Proof of Lemma 3.1. Starting from

$$p^* = p + \delta h(p) f(\zeta,p)^T A f(\tilde{\zeta},\tilde{p}) \Big(f(\zeta,p) - f(\tilde{\zeta},\tilde{p}) \Big) + r(q - \vec{1}/d) G(p),$$

we have, for any $i = 1,\ldots,d$,

$$\begin{aligned} p_i^* &\leq p_i + \delta h(p) + rG(p) \leq p_i + (1-p_i)(\delta + r) \leq 1, \\ p_i^* &\geq p_i - \delta h(p) - rG(p) \geq p_i(1 - \delta - r) \geq 0. \end{aligned}$$

To conclude the proof, let us show that $\sum_{i=1}^d p_i^* = 1$. Since $f(\zeta,p)$ and $f(\tilde{\zeta},\tilde{p})$ are vectors whose components are all equal to zero but one which is equal to one, we have

$$\sum_{i=1}^d \Big[f_i(\zeta,p) - f_i(\tilde{\zeta},\tilde{p}) \Big] = 0,$$

and, since $q \in \Delta$, we have

$$\sum_{i=1}^d (q_i - 1/d) = 0.$$

Then

$$\sum_{i=1}^{d} p_i^* = \sum_{i=1}^{d} p_i + \delta h f(\zeta)^T A f(\tilde{\zeta})\Big(\sum_{i=1}^{d} f_i(\zeta) - f_i(\tilde{\zeta})\Big) + rG(p)\sum_{i=1}^{d}(q_i - 1/d) = 1.$$

□

4. **A Boltzmann-like equation and its grazing limit.** We now consider an infinite population of agents interacting through the game defined by the matrix A and updating after each interaction their strategies according to the rule (3.4). We denote u_t the distribution of agents in the simplex of mixed strategies at time t. Notice that u_t is thus a probability measure on Δ. Intuitively, if this probability is regular enough, we can consider that u_t is its density, and $u_t(p)dp$ is roughly the proportion of agents whose strategy belongs to a neighborhood of volume dp around p.

4.1. **Boltzmann-type equation.** Let us find an equation describing the time evolution of u_t. However, since u_t is a measure, we only hope to find an equation in weak form, that is, for each observable $\int_\Delta \varphi(p)\,du_t(p)$, with $\varphi \in C(\Delta)$. Observe that this integral is the mean value at time t of some macroscopic quantity. For instance if $\varphi \equiv 1$ then $\int_\Delta \varphi(p)\,du_t(p) = u_t(\Delta)$ is the total mass of u_t, which should be conserved. If $\varphi(p) = p$ then $\int_\Delta \varphi(p)\,du_t(p) = \int_\Delta p\,du_t(p)$ is the mean strategy in the population. We will see later that it is strongly related to the replicator equation (2.2).

We can also think of u_t as the law of the stochastic process P_t giving the strategy of an arbitrary agent. Then $\int_\Delta \varphi(p)\,du_t(p) = \mathbb{E}[\varphi(P_t)]$ is the expected value of φ.

We assume that interactions take place following a unit rate Poisson process. Notice that if the Poisson process has constant rate, we can always assume that the rate is one up to a linear time re-scaling. Then it is standard to show that

$$\frac{d}{dt}\int_\Delta \varphi(p)\,du_t(p) = \int_{\Delta^2} \mathbb{E}[\varphi(p^*) - \varphi(p)]\,du_t(p)du_t(\tilde{p}), \tag{4.1}$$

see the book [33] for more details.

The following result states the existence and uniqueness of a solutions to this equation

Theorem 4.1. *For any initial condition $u_0 \in \mathcal{P}(\Delta)$ there exists a unique $u \in C([0,\infty),\mathcal{P}(\Delta)) \cap C^1((0,\infty),\mathcal{M}(\Delta))$ satisfying*

$$\int_\Delta \varphi(p)\,du_t(p) = \int_\Delta \varphi(p)\,du_0(p) + \int_0^t \int_{\Delta^2} \mathbb{E}[\varphi(p^*) - \varphi(p)]\,du_s(p)du_s(\tilde{p})ds$$

for any test-function $\varphi \in C(\Delta)$.

The proof of this result is classical and mainly based on Banach fixed-point theorem. It can also be proved viewing (4.1) as an ordinary differential equation in a Banach space following Bressan's insight [10] (see also [3]). For the reader convenience we provide the main steps of the proof in Appendix.

4.2. **Grazing limit.** We fix an initial condition $u_0 \in \mathcal{P}(\Delta)$ and denote u^δ the solution of (4.1) given by Theorem 4.1 corresponding to interaction rules (3.4). Notice that when $\delta \simeq 0$, $|p^* - p| \simeq 0$ so that $\varphi(p^*) - \varphi(p) \simeq (p^* - p)\varphi'(p) + \frac{1}{2}(p^* - p)^2\varphi''(p)$. Taking its expected value we thus obtain

$$\mathbb{E}[\varphi(p^*) - \varphi(p)] \simeq \mathbb{E}[p^* - p]\varphi'(p) + \frac{1}{2}\mathbb{E}[(p^* - p)^2]\varphi''(p).$$

Using the rules (3.4) to calculate the expectation and considering the new time scale $\tau = \delta t$, in the next theorem we obtain that we can approximate (4.1) when $\delta \simeq 0$ by a local equation of the form

$$\partial_\tau v + \operatorname{div}(\mathcal{F}[v]\, v) = \frac{\lambda}{2} \sum_{i,j=1}^{d} Q_{ij} \partial_{ij}(G^2\, v). \tag{4.2}$$

Here $\lambda := r^2/\delta$, the vector-field $\mathcal{F}[v]$ has components

$$\begin{aligned} \mathcal{F}_i[v_\tau](p) &= \sum_{k=1}^{d} h(p) a_{ik}(p_i \bar{p}_k(\tau) + p_k \bar{p}_i(\tau)) \\ &= h(p)(p_i e_i^T A \bar{p}(\tau) + \bar{p}_i(\tau) e_i^T A p), \end{aligned} \tag{4.3}$$

being

$$\bar{p}(\tau) = \int_\Delta p\, dv_\tau(p)$$

the mean-strategy at time τ. The diffusion coefficient Q_{ij} is the covariance of the uniform distribution on Δ, denoted θ, namely

$$Q_{ij} := \int_\Delta (q_i - 1/d)(q_j - 1/d)\, d\theta(q). \tag{4.4}$$

We say that $v \in C([0,\infty), \mathcal{P}(\Delta))$ is a weak solution to equation (4.2) if

$$\begin{aligned} &\int_\Delta \varphi(p)\, dv_t(p) - \int_\Delta \varphi(p)\, du_0 \\ &= \int_0^t \int_\Delta \nabla\varphi(p) \cdot \mathcal{F}[v_s](p)\, dv_s(p) + \frac{\lambda}{2} \sum_{i,j=1}^{d} Q_{ij} \int_\Delta \partial_{ij}\varphi(p) G^2(p)\, dv_s(p) ds \end{aligned} \tag{4.5}$$

for any $\varphi \in C^2(\Delta)$.

The above procedure is relatively well-known in the literature as grazing-limit. It has been introduced in the context of socio and econophysics modelling by Toscani [44], see also [33]. It can be rigourously justified to obtain the following Theorem:

Theorem 4.2. *Given an initial condition $u_0 \in \mathcal{P}(\Delta)$, let u^δ be the solution to equation (4.1) given by Theorem 4.1 corresponding to interaction rule (3.4).*

Assume that, as $\delta, r \to 0$, we have $r^2/\delta \to \lambda$. Let $\tau = \delta t$ and define $u_\tau^\delta := u_t^\delta$. Then there exists $v \in C([0,\infty), \mathcal{P}(\Delta))$ such that, as $\delta \to 0$ up to a subsequence, $u^\delta \to v$ in $C([0,T], \mathcal{P}(\Delta))$ for any $T > 0$. Moreover, v is a weak solution to equation (4.2) in the sense of (4.5).

Eventually, if $r^2/\delta^\alpha \to \lambda > 0$ for some $\alpha \in (0,1)$, then re-scaling time considering $\tau = \delta^\alpha t$, we obtain that $u^\delta \to v$ as before with v weak solution to

$$\frac{d}{d\tau} v = \frac{\lambda}{2} \sum_{i,j=1}^{d} Q_{ij} \partial_{ij}(G^2\, v).$$

On the other hand, if $r^2/\delta \to 0$, then re-scaling time we obtain that $u_\delta \to v$ with v a weak solution to

$$\partial_\tau v + div(\mathcal{F}[v]\, v) = 0. \tag{4.6}$$

In the rest of the paper we will focus in this last case, corresponding to the pure transport equation (4.6). Observe that this is a first order, nonlocal, mean field equation, and following a classic strategy going back at least to Neunzert and Wik [30], see also [9, 11, 19], we can prove directly the well-posedness of equation (4.6).

Given $v \in C([0,+\infty],\mathcal{P}(\Delta))$ we denote T_t^v the flow of the vector field $\mathcal{F}[v_t](x)$ namely

$$\frac{d}{dt}T_t^v(x) = \mathcal{F}[v_t](T_t^v(x)), \qquad T_{t=0}^v(x) = x.$$

It can be proved (see Appendix) that $T_t^v(x) \in \Delta$ for any $t \geq 0$. The result is the following:

Theorem 4.3. *For any initial condition $u_0 \in \mathcal{P}(\Delta)$, equation (4.6) has a unique weak solution u in $C([0,+\infty],\mathcal{P}(\Delta))$. This solution satisfies $u_t = T_t^v \sharp u_0$ for any $t \geq 0$.*

Moreover, the solutions depend continuously on the initial conditions. Indeed, there exists a continuous function $r : [0,+\infty) \to [0,+\infty)$ with $r(0) = 1$ such that for any pair of solutions $v^{(1)}$ and $v^{(2)}$ to equation (4.6) there holds

$$W_1\left(v_t^{(1)}, v_t^{(2)}\right) \leq r(t) W_1\left(v_0^{(1)}, v_0^{(2)}\right). \tag{4.7}$$

The proofs of Theorems 4.2 and 4.3 can be found in the Appendix.

5. Relationships between the mean-field equation, the replicator equations, and the game.

In this section we study the relationships between:

- solutions $v \in C([0,+\infty],\mathcal{P}(\Delta))$ to the mean-field equation (4.6), or in weak form,

$$\frac{d}{dt}\int_\Delta \varphi\, dv_t = \int_\Delta \mathcal{F}[v]\cdot\nabla\varphi\, dv_t \qquad \text{for any } \varphi \in C(\Delta), \tag{5.1}$$

  where the vector-field $\mathcal{F}[v]$ is given by (4.3),
- solutions to the replicator equations (2.2)

$$\frac{d}{dt}p_i = p_i((Ap)_i - p^T Ap) \qquad i = 1,\dots,d,$$

- Nash equilibria of the symmetric zero-sum game with pay-off matrix A.

We first relate the stationary weak solution to the mean-field equation (4.6) of the form δ_q with q an interior point and the Nash equilibria of the game. Indeed, we will prove that δ_q is a stationary weak solution if and only if q is a Nash equilibrium.

We then show that the mean strategy of the population satisfies the replicator equations. Finally, we will study the case of a two-strategies game, where we can precisely describe the asymptotic behavior of v_t, and then we show that, for generalizations of the classical Rock-Paper-Scissor where the trajectories of the replicator equations are closed orbits, v_t is also periodic.

These results are examples of the same heuristic principle: since v_t is obtained as the push-forward of v_0 by the flow of $\mathcal{F}[v_t]$ which is very much related to the replicator equation (at least, away from the boundary where $h = c$), the time evolution of v_t should be a consequence of the time evolution of the solutions to the replicator equation.

5.1. Nash equilibria and stationary solutions.

Given some probability measure $v \in \mathcal{P}(\Delta)$, we will slightly abuse notation considering v as the time-continuous function from $[0,+\infty)$ to Δ constantly equal to v.

Definition 5.1. We say that $v \in \mathcal{P}(\Delta)$ is an equilibrium or stationary solution of the transport equation (5.1) if it is a solution of (5.1), that is, if

$$\int_{\Delta} \mathcal{F}[v](p) \cdot \nabla\varphi(p)\, dv(p) = 0 \qquad \text{for all } \varphi \in C(\Delta). \tag{5.2}$$

We will mainly be interested in the case of equilibrium of the form $v = \delta_q$ for some interior point $q \in \Delta$.

Theorem 5.1. *Let q be an interior point of Δ. The following statements are equivalent:*

1. *q is an equilibrium of the replicator equations (2.2),*
2. *δ_q is an equilibrium of equation (5.1) in the sense of definition 5.1,*
3. *q belongs to the null space of the matrix A,*
4. *q is a Nash equilibrium of the symmetric zero-sum game with pay-off matrix A.*

Proof of Theorem 5.1. Let us first rewrite condition (5.2) for δ_q to be an equilibrium of equation (5.1). First notice that, for $v = \delta_q$, the mean strategy is obviously q. Then from definition (4.3) of the vector-field $\mathcal{F}_i[\delta_q]$ we have for any $p \in \Delta$ and any time $t \geq 0$ that

$$\mathcal{F}_i[\delta_q](p,t) = \sum_{k=1}^{d} h(p) a_{ik}(p_i q_k + p_k q_i).$$

In particular with $p = q$ we have

$$\mathcal{F}_i[\delta_q](q,t) = \sum_{k=1}^{d} h(q) a_{ik}(q_i q_k + q_k q_i) = 2h(q) q_i e_i^t A q = 2h(q) q_i (e_i^t A q - q^T A q)$$

where we used that A is antisymmetric so that $q^T A q = 0$. We can now rewrite condition (5.2). Notice that

$$\int_{\Delta} \mathcal{F}[\delta_q] \cdot \nabla\varphi\, d\delta_q = \mathcal{F}[\delta_q](q) \cdot \nabla\varphi(q) = 2h(q) \sum_{i=1}^{d} q_i (e_i^t A q - q^T A q) \partial_i \varphi(q).$$

Recall that q is an interior point of Δ so that $h(q) \neq 0$. We thus obtain that δ_q is an equilibrium of equation (5.1) if and only if

$$\sum_{i=1}^{d} q_i (e_i^t A q - q^T A q) \partial_i \varphi(q) = 0 \qquad \text{for all } \varphi \in C(\Delta). \tag{5.3}$$

We can now easily prove that statements (1) and (2) are equivalent. Indeed if q is an equilibrium of the replicator equations then $q_i((Aq)_i - q^T A q) = 0$ for any $i = 1, \ldots, d$ and (5.3) holds. On the other hand if δ_q is an equilibrium of the equation (5.1), i.e., condition (5.3) holds for any $\varphi \in C(\Delta)$, then taking $\varphi(p) = p_i$ we obtain $q_i(e_i^t A q - q^T A q) = 0$ for any $i = 1, \ldots, d$ so we get that q is an equilibrium of the replicator equation.

We can also prove that (1) and (3) are equivalent. Indeed if (1) holds, then $q_i(e_i^t A q - q^T A q) = 0$ for any $i = 1, \ldots, d$. Since $q^T A q = 0$ and $q_i > 0$ for any $i = 1, \ldots, d$, being q an interior point, the previous equality can be rewritten as $e_i^t A q = 0$ for any $i = 1, \ldots, d$, i.e., $Aq = 0$. This prove that (1) implies (3). On the other hand, if $Aq = 0$, then $e_i^t A q = 0$ for any $i = 1, \ldots, d$ so we obtain that (5.3) holds.

It remains to show that $Aq = 0$ if and only if q is a Nash equilibrium. Suppose that $Aq = 0$. Then for any $p \in \Delta$,

$$p^T Aq = p \cdot \vec{0} = 0 = q^T Aq.$$

Thus, playing any other strategy than q against q does not increase the pay-off. This means that q is a Nash equilibrium. Let us now assume that q is a Nash equilibrium and let us prove that $(Aq)_i = 0$ for any $i = 1, \dots, d$. If $(Aq)_i > 0$ then $e_i^T Aq > 0 = q^T Aq$ contradicting that q is a Nash equilibrium. If $(Aq)_i < 0$ then recalling that A is antisymmetric,

$$0 = q^T Aq = \sum_{k=1}^{d} q_k (Aq)_k = q_i (Aq)_i + \sum_{k \neq i} q_k (Aq)_k.$$

Since $q_k > 0$ for any $k = 1, \dots, d$ there must exists some $l \in \{1, \dots, d\}$ such that $(Aq)_l > 0$ and this is not possible.

The proof is finished. $\square$

In view of the previous result it is tempting to prove that if an interior point $q \in \Delta$ is locally asymptotically stable, then δ_q is locally asymptotically stable for (5.1). We can actually show this is true at least if the linearized matrix at q of the replicator equation has negative eigenvalues (in which case the convergence of v_t to δ_q is asymptotically fast when close to q, but is slower near the boundary). Unfortunately a zero-sum symmetric game cannot have such an equilibrium point since the replicator dynamics flow preserves volume (see Theorem 9.1.5. in [39]). We are currently working on extending our model to general two-players game to include in particular games with such an equilibrium.

5.2. **Evolution of the mean strategy and the replicator equations.** The method of characteristic yields that the solution to equation (5.1) is

$$v_t = T_t \sharp v_0 \tag{5.4}$$

where T_t is the flow of the vector-field $\mathcal{F}[v_t](p)$. This vector-field is the same as the one in the replicator equations but with the mean-strategy $\bar{p}(t)$ depending on the distribution v_t of strategies. The next result shows that $\bar{p}(t)$ satisfies (up to a constant) the replicator equations.

Theorem 5.2. *Consider a solution $v \in C([0, \infty), \mathcal{P}(\Delta))$ of the transport equation (5.1) staying away from the boundary $\partial\Delta$ of Δ up to some time $T > 0$, i.e.,*

$$dist\Big(supp(v_t), \partial\Delta\Big) \geq c^{1/d} \qquad 0 \leq t \leq T, \tag{5.5}$$

where c is defined in (3.1). Then the mean strategy $\bar{p}(t) = \int_\Delta p \, dv_t(p)$ is a solution of the replicator equation:

$$\frac{d}{dt}\bar{p}_i(t) = 2c\bar{p}_i(t) e_i^T A\bar{p}(t) \qquad i = 1, \dots, d. \tag{5.6}$$

Notice that (5.6) is not exactly the replicator systems due to the constant $2c$, but becomes so after the time-scale change $\tau = 2ct$.

Remark 5.1. Since v_t is the push-forward of v_0 by the flow of the replicator equation away from the boundary, assumption (5.5) will hold whenever the trayectories of the replicator equation starting from point in the support of v_0 stay at distance at least $c^{1/d}$ of the boundary. This occurs for instance in the case of the Rock-Paper-Scissors game since the orbits of the replicator system are closed curves circling around $(1/3, 1/3, 1/3)$. We refer to Remak 5.2 below.

Proof of Theorem 5.2. Let $\mathcal{T}_{s,t}$ be the flow of the vector-field $\mathcal{F}[v](t,x)$, i.e.,

$$\frac{d}{dt}\mathcal{T}_{s,t}(p) = \mathcal{F}[v](\mathcal{T}_{s,t}(p), t), \qquad \mathcal{T}_{s,s}(p) = p.$$

We also let $\mathcal{T}_t(p) = \mathcal{T}_{0,t}(p)$ and denote $\mathcal{T}_t^i(p)$, $i = 1, \ldots, d$, its components. Then $v_t = \mathcal{T}_t \sharp v_0$. It follows that for any $i = 1, \ldots, d$,

$$\bar{p}_i(t) = \int_\Delta p_i \, dv_t(p) = \int_\Delta \mathcal{T}_t^i(p) \, dv_0(p),$$

so that

$$\begin{aligned}
\frac{d}{dt}\bar{p}_i &= \int_\Delta \frac{d}{dt}\mathcal{T}_t^i(p) \, dv_0(p) = \int_\Delta \mathcal{F}_i[v_t](\mathcal{T}_t(p)) \, dv_0(p) \\
&= \sum_{k=1}^d \int_\Delta h\Big(\mathcal{T}_t(q)\Big) a_{ik} \Big(\mathcal{T}_t^i(q)\bar{p}_k(t) + \mathcal{T}_t^k(q)\bar{p}_i(t)\Big) \, dv_0(q) \\
&= \sum_{k=1}^d \int_\Delta h(p) a_{ik} (p_i \bar{p}_k(t) + p_k \bar{p}_i(t)) \, dv_t(p),
\end{aligned}$$

where we used once again that $v_t = \mathcal{T}_t \sharp v_0$. According to assumption (5.5) we have for any p in the support of v_t that $p_i \geq c^{1/d}$ for $i = 1, \ldots, d$, which implies that $h(p) = c$. Thus,

$$\begin{aligned}
\frac{1}{c}\frac{d}{dt}\bar{p}_i(t) =& \sum_{k=1}^d \int_\Delta a_{ik}(p_i \bar{p}_k(t) + p_k \bar{p}_i(t)) \, dv_t(p) \\
=& 2\sum_{k=1}^d a_{ik} \bar{p}_i(t) \bar{p}_k(t)
\end{aligned}$$

which is (5.6). □

5.3. **Two strategies games.** Let us consider the case of a symmetric game with two strategies. The pay-off matrix A is then

$$A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

for some $b \in \mathbb{R}$. Notice that if $b > 0$ (resp. $b < 0$) then the first (respectively, second) strategy strictly dominates the other. We thus expect that all agents end up playing the dominating strategy except those initially playing exclusively the loosing strategy since they cannot move due to the presence of h in the interaction rule.

Let v_t be the weak solution to the transport equation (4.6) and μ_t be the distribution of p_1 (i.e., μ_t is the first marginal of v_t). This means that $v_t(A \times [0,1]) = \mu_t(A)$ for any Borel set $A \subset [0,1]$, which can be rewritten as

$$\iint \varphi(p_1) \, dv_t(p_1, p_2) = \int \varphi(p_1) \, d\mu_t(p_1)$$

for any measurable non-negative function $\varphi : [0,1] \to \mathbb{R}$.

Theorem 5.3. *Assume that $b > 0$ and write the initial condition as*

$$\mu_0 = (1-a)\delta_0 + a\tilde{\mu}_0$$

where $a \in [0,1]$ and $\tilde{\mu}$ is a probablity measure on $[0,1]$ such that $\tilde{\mu}_0(\{0\}) = 0$. Then

$$\lim_{t\to+\infty} \mu_t = (1-a)\delta_0 + a\delta_1.$$

In the same way, if $b < 0$ and $\mu_0 = (1-a)\delta_1 + a\tilde{\mu}_0$ where $\tilde{\mu}_0(\{1\}) = 0$ and $a \in [0,1]$, then $\mu_t \to (1-a)\delta_1 + a\delta_0$ as $t \to +\infty$.

Proof of Theorem 5.3. Let us assume that $b > 0$ (the proof when $b < 0$ is completely analogous). It follows from (4.6) that μ_t satisfies the following equation: for any $\varphi \in C^1([0,1])$,

$$\frac{1}{b}\frac{d}{dt}\int_0^1 \varphi\, d\mu_t = \int_0^1 \varphi'(p_1) v[\mu_t](p_1)\, d\mu_t(p_1), \tag{5.7}$$

where for any probability measure μ on $[0,1]$ the vector-field $v[\mu]$ is defined as

$$v[\mu](p_1) = \underbrace{\min\{p_1(1-p_1), c\}}_{h(p_1)}(p_1 + \bar{p}_1 - 2p_1\bar{p}_1), \tag{5.8}$$

where

$$\bar{p}_1 = \int_0^1 p_1\, d\mu(p_1).$$

Notice first that since $p_1, \bar{p}_1 \in [0,1]$ we have

$$v[\mu](p_1) \geq h(p_1)(p_1^2 + \bar{p}_1^2 - 2p_1\bar{p}_1) = h(p_1)(p_1 - \bar{p}_1)^2. \tag{5.9}$$

Hence, it follows that $v[\mu] \geq 0$. Also, we claim that

$$v[\mu] = 0 \ \mu\text{-a.e.} \qquad \Leftrightarrow \qquad \mu = \alpha\delta_0 + (1-\alpha)\delta_1,\ \alpha \in [0,1]. \tag{5.10}$$

Indeed, if $v[\mu](p) = 0$ then $p_1 = 0$, $\bar{p}_1$ or 1 by (5.9). Thus if $v[\mu] = 0$ μ-a.e. then $\mu = \alpha\delta_0 + \beta\delta_1 + \gamma\delta_k$ for some $k \in (0,1)$ with $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma = 1$. Since $h(k) \neq 0$, $v[\mu](k) = 0$ gives $k = \bar{p}_1$ by (5.9) and then $0 = v[\mu](k) = h(k)(k + k - 2k^2)$ i.e., $k = 0$ or $k = 1$ which is an absurd.

Let us recall that $\mu_t = T_t \sharp \mu_0$ where T_t is the flow of $v[\mu_t](p_1)$, and also that $v[\mu_t](p_1) \geq 0$. Thus for any x in the support of μ_0, $T_t(x)$ is non-decreasing and bounded by 1 and thus converge to some $T_\infty(x)$. Then $\mu_t \to \mu_\infty := T_\infty \sharp \mu_0$.

Moroever, $v[\mu_\infty] = 0$ μ_∞-a.e. and thus $\mu_\infty = (1-\alpha)\delta_0 + \alpha\delta_1$ for some $\alpha \in [0,1]$ by (5.10).

To conclude, we have to show that

$$\mu_\infty(\{0\}) = \mu_0(\{0\}). \tag{5.11}$$

Let us take a smooth non-increasing function $\varphi : [0,1] \to [0,1]$ such that $\varphi = 1$ in $[0, 1/n]$ and $\varphi = 0$ in $[2/n, 1]$, $n \in \mathbb{N}$. Then

$$\int_0^1 \varphi\, d\mu_t - \int_0^1 \varphi\, d\mu_0 = \int_0^t \int_0^1 \varphi'(p_1) v[\mu_s](p_1)\, d\mu_s(p_1) ds \leq 0.$$

Letting $t = t_k \to +\infty$ we obtain

$$\int_0^1 \varphi\, d\mu_\infty \leq \int_0^1 \varphi\, d\mu_0$$

and then $\mu_\infty([0,1/n]) \le \mu_0([0,2/n])$. Letting $n \to +\infty$ gives $\mu_\infty(\{0\}) \le \mu_0(\{0\})$. To prove the converse inequality recall that $\mu_t = T_t \sharp \mu_0$ where T_t is the flow of $v[\mu_t](p_1)$. For any $\varphi \in C([0,1])$ we thus have

$$\int_0^1 \varphi\, d\mu_t = \int_0^1 \varphi(T_t(p_1))\, d\mu_0(p_1) = (1-a)\varphi(0) + a\int_0^1 \varphi(T_t(p_1))\, d\tilde{\mu}_0(p_1).$$

Letting $t = t_k \to +\infty$ we obtain $\int_0^1 \varphi\, d\mu_\infty \ge (1-a)\varphi(0)$ for any non-negative and continuous function φ. We deduce that $\mu_\infty(\{0\}) \ge 1-a$. This proves (5.11).

We conclude that $\mu_t \to (1-a)\delta_0 + a\delta_1$, and this finishes the proof. □

Given an initial condition μ_0, the distribution μ_t of p_1 is the unique weak solution (see (5.7)) of

$$\frac{1}{b}\partial_t \mu_t + \partial_{p_1}\Big(v[\mu_t](p_1)\Big) = 0.$$

In particular if μ_0 is a convex combination of Dirac masses like e.g. $\mu_0 = \frac{1}{N}\sum_{i=1}^N \delta_{p_1^i(0)}$ with $p_1^1(0), \dots, p_1^N(0) \in [0,1]$, then $\mu_t = \frac{1}{N}\sum_{i=1}^N \delta_{p_1^i(t)}$ where $p_1^1(t), \dots, p_1^N(t)$ are the solutions of the system

$$\begin{aligned}\frac{1}{b}\frac{d}{dt}p_1^i(t) &= v[\mu](p_1^i(t)) \\ &= \min\left\{p_1^i(1-p_1^i), c\right\}(p_1^i(t) + \bar{p}_1 - 2p_1^i(t)\bar{p}_1) \qquad i = 1,\dots,N,\end{aligned} \tag{5.12}$$

where

$$\bar{p}_1 = \frac{1}{N}\sum_{i=1}^N p_1^i(t).$$

We solved numerically the system (5.12) in the time interval $[0,T]$ using a Runge-Kutta scheme of order 4 with step size $h = 0.1$ with the following parameters values:

$$b = 1, \qquad c = 0.01, \qquad N = 1000, \qquad T = 800, \tag{5.13}$$

and taking as initial condition

$$p_1^k(0),\ k = 1,\dots,N, \text{ uniformly and independently distributed in } [0,0.3]. \tag{5.14}$$

We do not consider Dirac masses at $p = 0$ since we know from (5.12) that such masses would remain forever at $p = 0$. We show in Figure 1 (left) the resulting evolution of the distributions of the $p_1^k(t)$, $k = 1,\dots,N$. We can see that they are moving to the right until reaching 1 thus building up progressively the Dirac mass δ_1 in complete agreement with Theorem 5.3.

To illustrate numerically the conclusion of Theorem 4.2, we simulated the agent-based model by considering a population of $N = 1000$ agents interacting through the game defined by the pay-off matrix with $b = 1$ and updating their mixed strategy following the interaction rule (3.4). For a 2-strategies game, a mixed strategy is given by the probability p_1 of playing the first pure strategy (which is the dominating strategy here since $b > 0$). In that particular case, interaction rule (3.4) simplifies to

$$p_1^* = p_1 + \delta h(p_1) + r(q - 1/2)h(p_1)$$

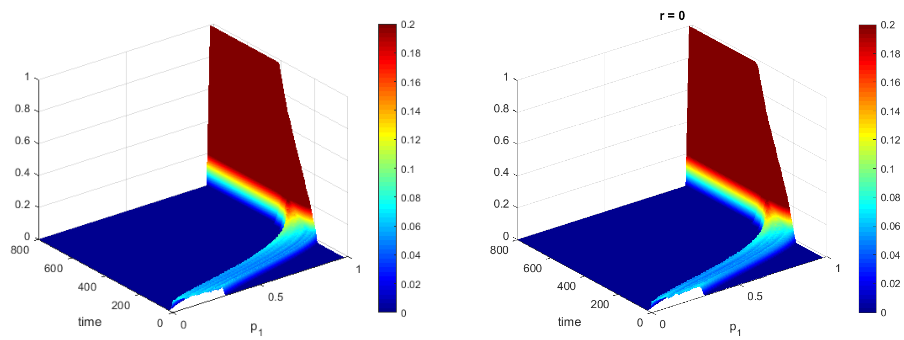
when the two agents involved in the interaction play different strategy, and

$$p_1^* = p_1 + r(q - 1/2)h(p_1)$$

otherwise. In the numerical experiments we chose $G(p_1) = h(p_1)$ with $c = 0.01$, and $\delta = 0.01$. Denoting $p_1^1, .., p_1^N$ the p_1 of the N agents, we chose them initially independently and uniformly distributed in $[0, 0.3]$ as in (5.14). At each time slot t

we make each agent interact once. To be coherent with the time scale τ, a τ time slot corresponds to $1/\delta = 100$ t-time slots. We plot in Figure 1 (right) the resulting evolution of the distribution of $p_1^1, \dots, p_1^N$ in the time time scale τ in absence of noise ($r = 0$). Notice that both images in Figure 1 are almost identical, in agreement with the conclusions of Theorem 4.2 when $r = 0$.

FIGURE 1. Time evolution of the distribution of $p_1^1, \dots, p_1^N$ obtained solving the transport equation (5.12) (left) and from an agent-based simulation (right).



We do not intend here to numerically solve the Fokker-Planck equation (4.2) which is a non-trivial task since the diffusion degenerates at the boundary. Instead, we present in Figure 2 some agent-based simulations to illustrate the influence of the noise level r on the dynamic. The parameters and initial condition are the same as before, and the noise level r is changed. These simulations suggest the existence of threshold value r^* for the noise level r such that for low noise level $r < r^*$ the distribution converges as $t \to +\infty$ to δ_1 whereas for high noise level $r > r^*$ it converges to a convex combination of δ_0 and δ_1, and a positive fraction of agents end up playing always the dominated strategy. To confirm this intuition we show in Figure 3 the averaged proportion of agents with $p \in [0, 0.01]$ at time $\tau = 800$ (the average is computed over 10 simulations). We clearly see a critical noise level value $r^* \simeq 0.41$ such if $r < r^*$ then all agents end-up playing the dominating strategy, and if $r > r^*$ a positive fraction of agents learn to play the dominated strategy. Thus a high noise level can lead to the learning of inefficient strategies. We know that when $r^2 \gg \delta$ the Fokker-Planck equation (4.2) (with $G = h$) for the distribution μ_t of p_1 becomes the purely diffusive equation

$$\partial_\tau \mu_\tau = \frac{\lambda}{2} Q_{11} \partial_{p_1 p_1}(h^2(p_1)\mu_\tau).$$

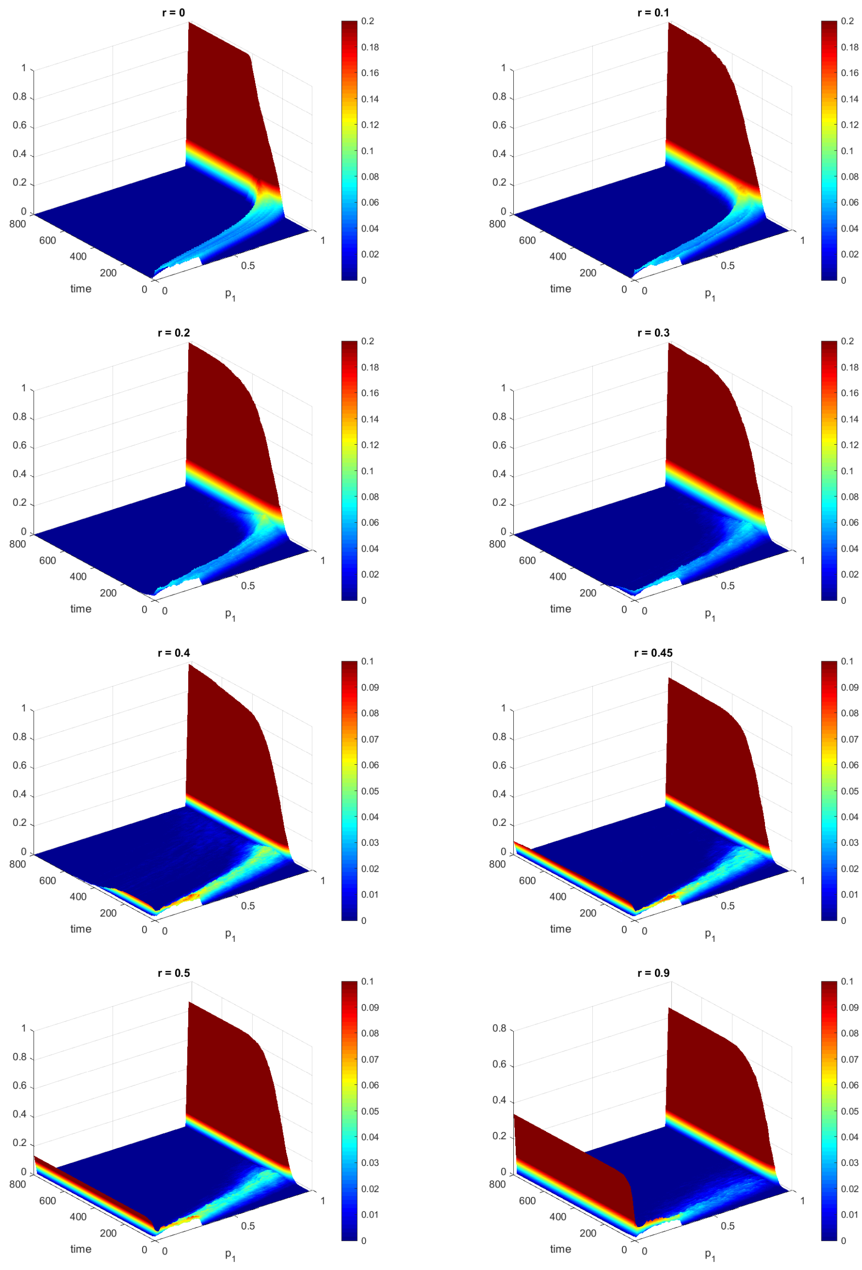
The convergence of μ_t as $t \to +\infty$ toward a convex combination $a\delta_1 + (1-a)\delta_0$ could then follow as in [41] where the authors proved a similar result for the equation

$$\partial_t P(t,x) = \partial_{xx}((1-x^2)P(t,x)) \qquad x \in [-1,1].$$

Notice however that the situation here is more delicate since the simulations shown in Figure 2 clearly indicate that the transport term is not negligeable. We leave these questions to future works.

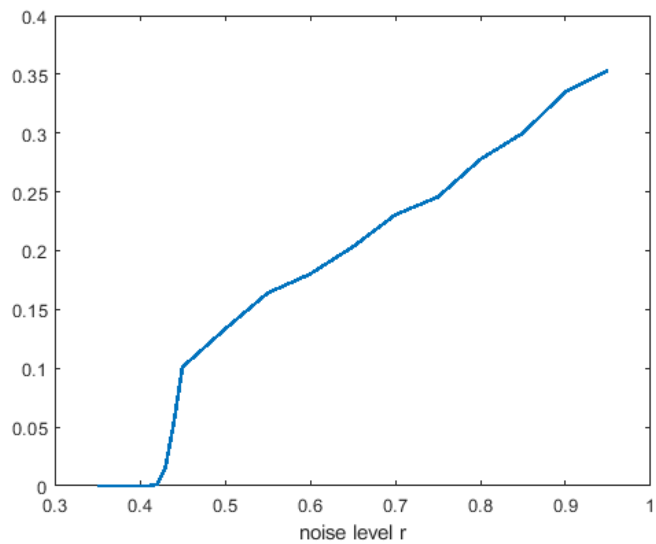
We eventually comment on the influence of the function h which was introduced to slow down the dynamic near the boundary of Δ and ensure that the

FIGURE 2. Time evolution of the distribution of $p_1^1, \ldots, p_1^N$, $N = 1000$, in the agent-based simulation starting from a uniform distribution in $[0, 0.3]$ with $\delta = 0.01$ and different values of noise level r.



post-interaction strategies $p^*, \tilde{p}^*$ in (3.4) remain in Δ. Since h is a constant far away from the boundary, it only impacts on the dynamic near the boundary. For instance, taking $\tilde{h}(x) = \min\{(x(1-x))^2, c\}$ instead of $h(x) = \min\{x(1-x), c\}$ does

FIGURE 3. Averaged proportion of agents in the agent-based simulation with $p \in [0, 0.01]$ at time $\tau = 800$ in function of the noise level r (the average is computed over 10 simulations).



not change the convergence toward the dominating strategy but only its speed. We illustrate this in Figure 4 showing the time evolution of the logarithm of the convex hull of the distribution of p_1, namely

$$\log(\max p_1 - \min p_1)$$

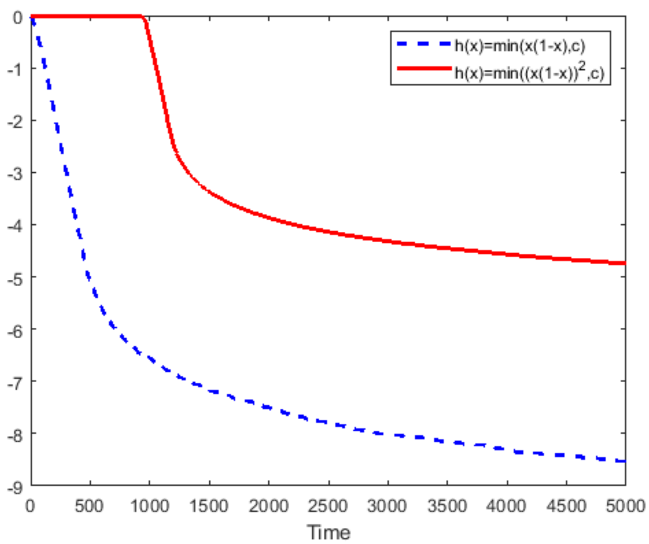
where the max and min are taken over the entire population, during a simulation of the agent-based model with h and $\tilde{h}$, starting from a distribution of p_1 uniform in $[0, 1]$, with $r = 0$, $\delta = 0.01$, $c = 0.01$. We can appreciate that the convergence toward δ_1 is much faster with h than with $\tilde{h}$. This is due to the fact that an agent with low or high p_1 moves much faster with h that $\tilde{h}$. Notice in particular that for $\tilde{h}$, $\max p_1 - \min p_1 \simeq 1$ up to time $t \simeq 1000$ due to the fact that an agent with initial $p \simeq 0$ move toward 1 very slowly.

5.4. **Periodic solutions for Paper-Rock-Scissors like games.** We now consider the case of a game for which the solutions of the replicator equations are periodic orbits. We have in mind the classic Rock-Paper-Scissors game whose payoff matrix is

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

In this game strategies dominate each other in a cyclic way as $1 \to 2 \to 3 \to 1$. This can be generalized to a game with an odd number d of strategies considering the pay-off matrix

FIGURE 4. Plot of $\log(\max p_1 - \min p_1)$ in the agent-based simulation with $N = 1000$, $\delta = c = 0.01$, $r = 0$ and two different functions h.



$$A := \begin{pmatrix} 0 & a_1 & a_2 & \cdots & \cdots & a_{d-1} \\ a_{d-1} & 0 & a_1 & a_2 & \cdots & a_{d-2} \\ a_{d-2} & a_{d-1} & 0 & a_1 & \cdots & a_{d-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & \cdots & a_{d-1} & 0 \end{pmatrix}$$

with $a_k = (-1)^{k-1}$, $k = 1, \dots, d-1$.

It is known that interior trajectories for the replicator equations for these games are closed periodic orbits enclosing the unique interior Nash equilibrium $N := \frac{1}{d}\vec{1}$ (see e.g. [39]). In particular

$$N^T A N = 0.$$

Moreover (see Theorem 7.6.4 in [23]) if $p(t)$ is such a trajectory then its temporal mean converges to N:

$$lim_{t\to+\infty} \frac{1}{t}\int_0^t p(s)\,ds = N.$$

It follows that

$$\frac{1}{T}\int_0^T p(s)\,ds = N \tag{5.15}$$

where T is the period of the trajectory $x(t)$.

Our next result states that the weak solutions of the mean-field equation (4.6) lying in $\{h = c\}$ are also periodic.

Theorem 5.4. *Consider a weak solution v_t to equation (4.6) with initial condition v_0 such that v_t is supported in $\{h = c\}$ for any $t \geq 0$. If the initial mean strategy is different from N then there exists $T > 0$ such that $v_{t+T} = v_t$ for any $t \geq 0$.*

Remark 5.2. Let us note that, given a ball B centered at N, any trajectory of the replicator equations starting at some $p \in B \cap \{p_1 + \cdots + p_d = 1\}$ cannot reach the boundary of Δ, since whenever a coordinate of $p(t)$ is equal to zero, it remains zero for every t. So, choosing B sufficiently small, all the trajectories remain in the set $\{h = c\}$.

Assuming that $supp(v_0) \subset B$, we get that $supp(v_t) \subset \{h = c\}$.

Proof of Theorem 5.4. Consider a weak solution v_t to equation (4.6) with initial condition v_0. Then $v_t = \mathcal{T}_t \sharp v_0$ where $\mathcal{T}_t$ is the flow of $\mathcal{F}[v_t](p)$. Thus to prove that v_t is periodic, it is enough to prove that all the trayectories $t \to \mathcal{T}_t(p)$, $p \in \operatorname{supp}(v_0)$, are periodic with the same period. Let $p(t) = \mathcal{T}_t(p)$ be such a trajectory. Since $h(p(t)) = c$ for any t,

$$\begin{aligned} \frac{d}{dt}p(t) &= \mathcal{F}[v_t](p(t)) \\ &= c(B(t) + C(t)A)p(t) \end{aligned}$$

where

$$B(t) = diag((Am(t))_1, \ldots, (Am(t))_d), \qquad C(t) = diag(m_1(t), \ldots, m_d(t)),$$
$$m(t) = \int_\Delta p\, dv_t(p).$$

Thus

$$p(t) = exp\Big(c\int_0^t B(s) + C(s)A\, ds\Big)p(0).$$

According to Theorem 5.2, m is a solution to the replicator equations for A and thus is periodic. We denote its period by T. By (5.15) we deduce that

$$\frac{1}{T}\int_0^T m(s)\, ds = N.$$

Thus

$$\frac{1}{T}\int_0^T Am(t)\, ds = A\Big(\frac{1}{T}\int_0^T m(s)\, ds\Big) = AN = 0.$$

Recalling that $N = \frac{1}{d}\vec{1}$,

$$\frac{1}{T}\int_0^T B(s) + C(s)A\, ds = \frac{1}{T}\int_0^T C(s)\, ds.A = \frac{1}{d}A.$$

We deduce that

$$p(T) = exp\Big(\frac{cT}{d}A\Big)p(0).$$

The matrix $R := exp\Big(\frac{cT}{d}A\Big)$ is orthogonal (being A antisymmetric, $exp(A^t) = exp(-A) = (expA)^{-1}$) and has determinant $exp\Big(Tr(\frac{cT}{d}A)\Big) = 1$. Thus $R \in SO(d)$.

Moreover, $RN = (Id + \frac{cT}{d}A + ...)N = N$ so we get that R is a rotation around the line (ON). Since this line is perpendicular to the plane $\{p_1 + \cdots + p_d = 1\}$, the matrix R is a rotation in this plane fixing N.

We thus obtain that any trajectory $p(t)$ starting from $p(0) \in \operatorname{supp}(v_0)$ satisfies

$$p(T) = Rp(0)$$

where T is the period of m and R is a rotation in $\{p_1 + \cdots + p_d = 1\}$ fixing N. It follows in particular that $v_T = R\sharp v_0$ and $m(T) = Rm(0)$. Since $m(T) = m(0)$ by definition of T, we obtain that $m(0)$ is a another fixed-point of R. Thus if $N \neq m(0)$

then $R = Id$ so that $v_T = v_0$. Since the weak solution to the mean-field equation (4.6) with a given intial condition is unique, we deduce that $v_{t+T} = v_t$ for any $t \geq 0$.

The proof is finished $\square$

6. Appendix.

6.1. Existence of solution to the Boltzmann-like equation: proof of Theorem 4.1.

Given an initial condition $u_0 \in \mathcal{P}(\Delta)$ we want to prove that there exists a unique $u \in C([0,\infty), \mathcal{P}(\Delta))$ such that

$$\int_\Delta \varphi(p)\, du_t(p) = \int_\Delta \varphi(p)\, du_0(p) + \int_0^t \int_{\Delta^2} \mathbb{E}[\varphi(p^*) - \varphi(p)]\, du_s(p) du_s(\tilde{p}) ds \qquad (6.1)$$

for any $\varphi \in C(\Delta)$.

We split the proof into two steps.

Step 1. There is a unique $u \in C([0,\infty), \mathcal{M}(\Delta))$ satisfying (6.1).

Recall that $\mathcal{M}(\Delta)$ denotes the space of finite Borel measures on Δ that we endow, unless otherwise stated, with the total-variation norm.

Proof. Given $u, v \in \mathcal{M}(\Delta)$ we define a finite measure $Q(u,v)$ on Δ by

$$\begin{aligned}&\langle Q(u,v), \varphi\rangle \\ &:= \frac{1}{2}\int_{\Delta^3} \mathbb{E}[\varphi(p^*) - \varphi(p)]\, du(p) dv(\tilde{p}) + \frac{1}{2}\int_{\Delta^3} \mathbb{E}[\varphi(p^*) - \varphi(p)]\, dv(p) du(\tilde{p})\end{aligned}$$

for $\varphi \in C(\Delta)$. We also let $Q(u) := Q(u,u)$. For $u \in C([0,+\infty], \mathcal{M}(\Delta))$ we then define a map $J(u) : [0,+\infty) \to \mathcal{M}(\Delta)$ by

$$J(u)_t := u_0 + \int_0^t Q(u_s)\, ds, \qquad t \geq 0,$$

that is,

$$(J(u)_t, \varphi) := (u_0, \varphi) + \int_0^t (Q(u_s), \varphi)\, ds \qquad \text{for any } \varphi \in C(\Delta).$$

We thus look for a fixed point of J in $C([0,+\infty], \mathcal{M}(\Delta))$. We will apply Banach fixed-point theorem to J in the complete metric space

$$\mathcal{A} := \{u \in C([0,T], \mathcal{M}(\Delta)) :\, u(0) = u_0 \text{ and } \max_{0\leq s\leq T} \|u_s\|_{TV} \leq 2\}$$

where $T \in (0, 1/8)$.

Let us verify that $J(\mathcal{A}) \subseteq \mathcal{A}$. First notice that

$$\|Q(u,v)\|_{TV} \leq 2\|u\|_{TV}\|v\|_{TV}. \qquad (6.2)$$

Moreover, Q is bilinear so that $Q(u) - Q(v) = Q(u+v, u-v)$ and then

$$\|Q(u) - Q(v)\|_{TV} \leq 2\|u+v\|\|u-v\|_{TV}. \qquad (6.3)$$

Now for any $u, v \in \mathcal{A}$, and any $t \in [0,T]$,

$$\begin{aligned}\|J(u)_t)\|_{TV} &\leq \|u_0\|_{TV} + \int_0^t \|Q(v_s)\|_{TV}\, ds \\ &\leq 1 + 2T \max_{0\leq s\leq T} \|v_s\|_{TV}^2 \\ &\leq 1 + 8T \leq 2.\end{aligned}$$

Moreover, for $0 \le s \le t \le T$,

$$\|J(u)_t - J(u)_s\|_{TV} \le \int_s^t \|Q(u_\tau)\|_{TV}\,d\tau \le 2\int_s^t \|u_\tau\|_{TV}^2\,d\tau \le 8|t-s|,$$

and we deduce the continuity of $J(u)_t$ in t. Thus $J(\mathcal{A}) \subset \mathcal{A}$.

Now, for any $u, v \in \mathcal{A}$, using (6.3),

$$\begin{aligned}\|J(u)_t - J(v)_t\| &\le \int_0^t \|Q(u_s) - Q(v_s)\|\,ds \\ &\le \int_0^t 2\|u_s + v_s\|\|u_s - v_s\|\,ds \\ &\le 8T\|u - v\|,\end{aligned}$$

so that

$$\|J(u) - J(v)\| \le 8T\|u - v\|.$$

Thus choosing $T < 1/8$, we deduce that J is a strict contraction from $\mathcal{A}$ to $\mathcal{A}$ and therefore, has a unique fixed-point. Repeating the argument on $[T, 2T], \dots$, we obtain a unique $u \in C([0,+\infty], \mathcal{M}(\Delta))$ satisfying (6.1). □

Notice that it is not a priori obvious that $J(u)_t \ge 0$ if $u_t \ge 0$, $t \ge 0$. We verify that $u_t \ge 0$ in an indirect way in the next step following ideas from [12].

Step 2. u_t is a probability measure on Δ for any $t \ge 0$ where u is given by the previous step.

Proof. We first verify that u_t is a non-negative measure for any $t \ge 0$ i.e.

$$(u_t, \varphi) \ge 0 \qquad \text{for any } \varphi \in C(\Delta),\ \varphi \ge 0.$$

Given $u, v \in \mathcal{M}(\Delta)$ we define the measures

$$Q_+(u, v) := \frac{1}{2}\int_{\Delta^3} \mathbb{E}[\varphi(p^*)]\,(du(p)dv(\tilde{p}) + dv(p)du(\tilde{p}))$$

and $Q_+(v) := Q_+(v, v)$. Notice that $Q_+(u, v)$ is non-negative if both u and v are non-negative. Moreover,

$$Q(u) = Q_+(u) - u \tag{6.4}$$

and

$$Q_+(u) \ge Q_+(v) \ge 0 \qquad \text{if } u \ge v \ge 0 \tag{6.5}$$

since $Q_+(u) - Q_+(v) = Q_+(u + v, u - v) \ge 0$.

The idea of the proof consists in finding $v \in C([0,\infty), \mathcal{P}(\Delta))$ (continuity with respect to the total variation norm) such that

$$v_t = e^{-t}u_0 + \int_0^t e^{s-t}Q_+(v_s)\,ds. \tag{6.6}$$

Indeed in that case using (6.4),

$$\begin{aligned}\frac{d}{dt}v_t &= -e^{-t}u_0 - \int_0^t e^{s-t}Q_+(v_s)\,ds + Q_+(v_t) \\ &= Q_+(v_t) - v_t \\ &= Q(v_t).\end{aligned}$$

Thus, v_t verifies (6.1) so that $v = u$ since u is the unique solution of (6.1).

To obtain v satisfying (6.6) we consider the sequence $v^{(n)} \in C([0,+\infty), P(\Delta))$, $n \in \mathbb{N}$, defined by $v^{(0)} := 0$ and

$$v_t^{(n)} := e^{-t} u_0 + \int_0^t e^{s-t} Q_+(v_s^{(n-1)})\, ds.$$

Recalling that $u_0 \geq 0$ and using (6.5) it is easily seen that $v_t^{(n)} \geq v_t^{(n-1)} \geq 0$. Also notice that

$$(v_t^{(1)}, 1) = e^{-t} + \int_0^t e^{s-t}(Q_+(u_0), 1)\, ds = (Q_+(u_0), 1) = (Q(u_0), 1) + (u_0, 1) = 1$$

where we used (6.4) and the fact that $(Q(u), 1) = 0$ for any $u \in \mathcal{M}(\Delta)$. Notice that the function $(v_t^{(n)}, 1)$ satisfies the ordinary differential equation

$$\begin{cases} \frac{d}{dt}(v_t^{(n)}, 1) = (Q_+(v_t^{(n-1)}), 1) - (v_t^{(n)}, 1) = (v_t^{(n-1)}, 1) - (v_t^{(n)}, 1), \\ (v_{t=0}^{(n)}, 1) = (u_0, 1) = 1. \end{cases}$$

We can then prove by induction that $(v_t^{(n)}, 1) = 1$ for any n and t. Thus for any $t \geq 0$, $(v_t^{(n)})_n$ is a non-decreasing sequence of probability measures on Δ. We can then define a probability measure v_t on Δ by

$$(v_t, \varphi) := \lim_{n \to +\infty} (v_t^{(n)}, \varphi) \qquad \varphi \in C(\Delta).$$

In fact the convergence of $v^{(n)}$ to v is uniform in $t \in [0, T]$ for any $T > 0$, and thus v is continuous in t. This follows from the Arzela-Ascoli thorem. Indeed, since $\|v_t^{(n)}\|_{TV} = 1$, we only need to prove that the sequence $(v_t^{(n)})_n$ is uniformly equicontinuous. We have

$$\begin{aligned} \|v_{t+h}^{(n)} - v_t^{(n)}\|_{TV} \quad \leq \quad & |e^{t+h} - e^t| \|u_0\|_{TV} + \int_t^{t+h} e^{s-(t+h)} \|Q_+(v_s^{(n-1)})\|_{TV}\, ds \\ & + \int_0^t |e^{s-(t+h)} - e^{s-t}| \|Q_+(v_s^{(n-1)})\|_{TV}\, ds. \end{aligned}$$

In view of (6.2) and recalling that $v_s^{(n-1)} \in \mathcal{P}(\Delta)$ we have $\|Q_+(v_s^{(n-1)})\|_{TV} \leq \|Q(v_s^{(n-1)})\|_{TV} + \|v_s^{(n-1)}\|_{TV} \leq 3$. The uniform equi-continuity follows easily.

This ends the proof. □

To conclude the proof of Theorem 4.1, we verify that $u \in C^1((0,+\infty), \mathcal{M}(\Delta))$ with $\partial_t u_t = Q(u_t)$. Indeed, recalling that $u_t = J(u)_t$, we have

$$\frac{u_{t+h} - u_t}{h} - Q(u_t) = \frac{1}{h}\int_t^{t+h} Q(u_s)\, ds - Q(u_t) = \frac{1}{h}\int_t^{t+h} Q(u_s) - Q(u_t)\, ds.$$

Using (6.3) together with $\|u_t\|_{TV} = 1$ we obtain

$$\begin{aligned} \left\| \frac{u_{t+h} - u_t}{h} - Q(u_t) \right\|_{TV} \leq & \frac{2}{h}\int_t^{t+h} \|u_s + u_t\|_{TV} \|u_s - u_t\|_{TV}\, ds \\ \leq & \frac{4}{h}\int_t^{t+h} \|u_s - u_t\|_{TV}\, ds \end{aligned}$$

which goes to 0 as $h \to 0$ by the Dominated Convergence Theorem since u_s is continuous in s for the total variation norm.

6.2. **Grazing limit: Proof of Theorem 4.2.** The proof consists in two main steps. First approximating the difference $\varphi(p^*) - \varphi(p)$ in the Boltzmann-like equation (4.1) by a second order Taylor expansion, we obtain that u^δ is an approximate solution of (4.2) in the sense of (4.5). Then we apply Arzela-Ascoli Theorem to deduce that a subsequence of the u_δ converges to a solution of (4.5).

Before beginning the proof we need the following lemma which gives the expected value of of $f(\zeta,p)^T Af(\tilde{\zeta},\tilde{p})\Big(f_i(\zeta,p) - f_i(\tilde{\zeta},\tilde{p})\Big)$ where the random vector $f(\zeta;p) = (f_1(\zeta;p),\dots,f_d(\zeta;p))$ is defined in (3.3).

Lemma 6.1. *For any $i,j = 1,\dots,d$, any $p,\tilde{p} \in \Delta$ and any independent random variables $\zeta,\tilde{\zeta}$ uniformly distributed in $[0,1]$, there hold*

$$\mathbb{E}\Big[f(\zeta,p)^T Af(\tilde{\zeta},\tilde{p})\Big(f_i(\zeta,p) - f_i(\tilde{\zeta},\tilde{p})\Big)\Big] = \sum_{k=1}^{d} a_{ik}(p_i\tilde{p}_k + p_k\tilde{p}_i). \tag{6.7}$$

Proof. Let us denote $f_i = f_i(\zeta,p)$ and $\tilde{f}_i = f_i(\tilde{\zeta},\tilde{p})$. The proof is based on the following two properties of the f_i:

$$f_i f_j = \delta_{ij} \qquad \text{and} \qquad f_i^2 = f_i,$$

which follows from the definition of $f(\zeta,p)$. We write

$$\begin{aligned} f(\zeta,p)^T Af(\tilde{\zeta},\tilde{p})\Big(f_i(\zeta,p) - f_i(\tilde{\zeta},\tilde{p})\Big) &= \sum_{m,n=1}^{d} a_{mn} f_m \tilde{f}_n \Big(f_i - \tilde{f}_i\Big) \\ &= \sum_{n=1}^{d} a_{in} f_i \tilde{f}_n - \sum_{m=1}^{d} a_{mi} f_m \tilde{f}_i. \end{aligned}$$

Since ζ and $\tilde{\zeta}$ are independent, so are $f_i(\zeta,p)$ and $f_j(\tilde{\zeta},\tilde{p})$ for any $i,j = 1,\dots,d$. Moreover $\mathbb{E}[f_i(\zeta,p)] = p_i$ since $f_i(\zeta,p) = 1$ with probability p_i and $f_i(\zeta,p) = 0$ with probability $1 - p_i$. Taking the expectation in (6.2) we thus obtain

$$\mathbb{E}\Big[f(\zeta,p)^T Af(\tilde{\zeta},\tilde{p})\Big(f_i(\zeta,p) - f_i(\tilde{\zeta},\tilde{p})\Big)\Big] = \sum_{n=1}^{d} a_{in} p_i \tilde{p}_m - \sum_{m=1}^{d} a_{mi} p_m \tilde{p}_i.$$

We deduce (6.7) recalling that $a_{ij} = -a_{ji}$ being A antisymmetric.

The proof is finished. ∎

We are now in position to prove Theorem 4.2. We split the proof into several steps. The first one states that u^δ is an approximate solution of (4.2) in the sense of (4.5).

Step 1. For any $\varphi \in C^3(\Delta)$,

$$\begin{aligned} &\int_\Delta \varphi(p)\, du_\tau^\delta(p) - \int_\Delta \varphi(p)\, du_0(p) \\ &= \int_0^t \int_\Delta \nabla\varphi(p)\mathcal{F}[u_s^\delta](p)du_s^\delta(p)ds + \frac{r^2}{2\delta}\int_0^t \int_\Delta \sum_{i,j=1}^{d} Q_{ij}\partial_{ij}\varphi(p)G(p)^2\, du_s^\delta(p)ds \\ &+ \int_0^t Error(s,\delta)ds \end{aligned} \tag{6.8}$$

where

$$|Error(s,\delta)| \le \frac{1}{\delta} C\|D^3\varphi\|_\infty(\delta^3 + r^3)2 + C\|D^2\varphi\|_\infty \delta, \tag{6.9}$$

and the constant C is independent of t, φ, δ and r.

Proof. First, for any test-function φ we have

$$\begin{aligned}\frac{d}{d\tau}\int_\Delta \varphi(p)\, du^\delta_\tau(p) =& \frac{1}{\delta}\frac{d}{dt}\int_\Delta \varphi(p)\, du^\delta(p,t)\\ =& \frac{1}{\delta}\int_{\Delta^2} \mathbb{E}\Big[\varphi(p^*) - \varphi(p)\Big] du^\delta_t(p) du^\delta_t(\tilde p).\end{aligned}$$

Performing a Taylor expansion up to the second order we have

$$\varphi(p^*) - \varphi(p) = \sum_{i=1}^d \partial_i\varphi(p)(p^*_i - p_i) + \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}\varphi(p)(p^*_i - p_i)(p^*_j - p_j) + R(p^*, p)$$

where

$$|R(p^*,p)| \le \frac{1}{6}\|D^3\varphi\|_\infty |p^* - p|^3. \tag{6.10}$$

Thus,

$$\begin{aligned}&\int_{\Delta^2} \mathbb{E}\Big[\varphi(p^*) - \varphi(p)\Big] du^\delta_t(p) du^\delta_t(\tilde p)\\ &\quad = \int_{\Delta^2} \sum_{i=1}^d \partial_i\varphi(p)\mathbb{E}\Big[p^*_i - p_i\Big]\\ &\quad\quad + \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}\varphi(p)\mathbb{E}\Big[(p^*_i - p_i)(p^*_j - p_j)\Big] du^\delta_t(p) du^\delta_t(\tilde p)\\ &\quad\quad + \int_{\Delta^2} \mathbb{E}\Big[R(p^*,p)\Big] du^\delta_t(p) du^\delta_t(\tilde p).\end{aligned} \tag{6.11}$$

We examine each term in the right hand side. In view of the interaction rule (3.4), namely

$$p^* - p = \delta h(p) f(\zeta,p)^T A f(\tilde\zeta,\tilde p)\Big(f(\zeta,p) - f(\tilde\zeta,\tilde p)\Big) + r(q - \vec{1}/d)G(p),$$

we have

$$\mathbb{E}[p^*_i - p_i] = \delta h(p)\mathbb{E}\Big[f(\zeta,p)^T A f(\tilde\zeta,\tilde p)\Big(f_i(\zeta,p) - f_i(\tilde\zeta,\tilde p)\Big)\Big] + \mathbb{E}[r(q_i - 1/d)G(p)].$$

If q is a random variable uniformly distributed on Δ, then $\mathbb{E}[q_i] = 1/d$. So, in view of (6.7) in the previous Lemma, we obtain

$$\mathbb{E}[p^*_i - p_i] = \delta h(p)\sum_{k=1}^d a_{ik}(p_i\tilde p_k + p_k\tilde p_i).$$

By integrating we get

$$\int_{\Delta^2} \sum_{i=1}^d \partial_i\varphi(p)\mathbb{E}\Big[p^*_i - p_i\Big] du^\delta_t(p) du^\delta_t(\tilde p)$$

$$
\begin{aligned}
&= \delta \int_\Delta h(p) \sum_{i=1}^d \partial_i\varphi(p) \sum_{k=1}^d a_{ik}(p_i\bar{p}_k(t) + p_k\bar{p}_i(t))du_t^\delta(p) \\
&= \delta \int_\Delta \nabla\varphi(p)\mathcal{F}[u_t](p)du_t^\delta(p),
\end{aligned} \tag{6.12}
$$

where the vector-field $\mathcal{F}$ is defined in (4.3).

We now study $\mathbb{E}\Big[(p_i^* - p_i)(p_j^* - p_j)\Big]$. In view of the interaction rule,

$$
\begin{aligned}
(p_i^* - p_i)(p_j^* - p_j) =& \Big[\delta h(p)f(\zeta)^T Af(\tilde{\zeta})\Big(f_i(\zeta) - f_i(\tilde{\zeta})\Big) + r(q_i - 1/d)G(p)\Big] \\
&\times \Big[\delta h(p)f(\zeta)^T Af(\tilde{\zeta})\Big(f_j(\zeta) - f_j(\tilde{\zeta})\Big) + r(q_j - 1/d)G(p)\Big] \\
=& \Big(\delta h(p)f(\zeta)^T Af(\tilde{\zeta})\Big)^2 \Big(f_i(\zeta) - f_i(\tilde{\zeta})\Big)\Big(f_j(\zeta) - f_j(\tilde{\zeta})\Big) \\
&+ \delta h(p)f(\zeta)^T Af(\tilde{\zeta})\Big(f_i(\zeta) - f_i(\tilde{\zeta})\Big)r(q_j - 1/d)G(p) \\
&+ \delta h(p)f(\zeta)^T Af(\tilde{\zeta})\Big(f_j(\zeta) - f_j(\tilde{\zeta})\Big)r(q_i - 1/d)G(p) \\
&+ r^2(q_i - 1/d)(q_j - 1/d)G(p)^2.
\end{aligned}
$$

The second and third term have zero expected value since q is independent of ζ and $\tilde{\zeta}$ and $\mathbb{E}[q_i] = 1/d$. Moreover, in view of the definition (4.4) of Q, the expectation of the last term is $r^2G(p)^2Q_{ij}$. Lastly, since $|f_i(\zeta, p)| \le 1$ for any $i = 1, \dots, d$ and any $p \in \Delta$, the expectation of the first term can be bounded by $C\delta^2$ for a constant C depending only on c and the coefficients of A. Thus

$$
\mathbb{E}\Big[(p_i^* - p_i)(p_j^* - p_j)\Big] = r^2G(p)^2Q_{ij} + O(\delta^2).
$$

By integrating,

$$
\begin{aligned}
&\int_{\Delta^2} \sum_{i,j=1}^d \partial_{ij}\varphi(p)\mathbb{E}\Big[(p_i^* - p_i)(p_j^* - p_j)\Big]du_t^\delta(p)du_t^\delta(\tilde{p}) \\
&= r^2 \int_{\Delta^2} G(p)^2 \sum_{i,j=1}^d \partial_{ij}\varphi(p)Q_{ij}du_t^\delta(p) + O(\delta^2)\|D^2\varphi\|_\infty.
\end{aligned} \tag{6.13}
$$

It remains to bound the error term

$$
\int_{\Delta^2} \mathbb{E}\Big[R(p^*, p)\Big]du_t^\delta(p)du_t^\delta(\tilde{p}) \le \frac{1}{6}\|D^3\varphi\|_\infty \int_{\Delta^2} \mathbb{E}\Big[|p^* - p|^3\Big]du_t^\delta(p)du_t^\delta(\tilde{p}).
$$

Using that $(a+b)^3 \le C(a^3 + b^3)$ for any $a, b \ge 0$, $|G(p)| \le 1$, $|h(p)| \le c$, and $|f(\zeta)^T Af(\tilde{\zeta})| \le \sum_{i,j=1}^d |a_{ij}|f_i(\zeta, p)f_i(\tilde{\zeta}p) \le \sum_{i,j=1}^d |a_{ij}|$, we have

$$
\begin{aligned}
|p^* - p|^3 &\le \Big|\delta h(p)f(\zeta, p)^T Af(\tilde{\zeta}, \tilde{p})\Big(f(\zeta, p) - f(\tilde{\zeta}, \tilde{p})\Big) + r(q - \vec{1}/d)G(p)\Big|^3 \\
&\le C\delta^3c^3\Big(\sum_{i,j=1}^d |a_{ij}|\Big)^3 + Cr^3 = C(\delta^3 + r^3).
\end{aligned}
$$

Thus,

$$
\int_{\Delta^2} \mathbb{E}\Big[R(p^*, p)\Big]du_t^\delta(p)du_t^\delta(\tilde{p}) \le C\|D^3\varphi\|_\infty(\delta^3 + r^3). \tag{6.14}
$$

By inserting (6.12), (6.13) and (6.14) into (6.11), we obtain (6.8). $\square$

We now verify that the sequence (u^δ) verifies the assumptions of Arzela-Ascoli Theorem, namely boundedness and uniform equicontinuity. The proof is based of the previous step. Since the error term (6.9) involves $\|\varphi\|_{C^3}$, the Wasserstein distance is of no use. Instead we will use the norm $\|u\|_{sup}$, $u \in \mathcal{P}(\Delta)$, defined by

$$\|u\|_{sup} := \sup_{\varphi} \int_\Delta \varphi(p)\, du(p) \tag{6.15}$$

where the supremum is taken over all the functions $\varphi \in C^3(\Delta)$ such that $\|\varphi\|_3 \leq 1$ where

$$\|\varphi\|_3 := \sum_{|\alpha|\leq 3} \|\partial^\alpha \varphi\|_\infty. \tag{6.16}$$

According to [20] this norm metricizes the weak convergence in $\mathcal{P}(\Delta)$.

Step 2. For any $t \in [0, T]$ and any $\delta > 0$,

$$\|u^\delta_\tau\|_{sup} \leq 1. \tag{6.17}$$

Moreover there exists a constant $K > 0$ such that for any $\tau, \tau' \in [0, T]$ and any $r, \delta > 0$ small,

$$\|u^\delta_\tau - u^\delta_{\tau'}\|_{sup} \leq K\, |\tau - \tau'|. \tag{6.18}$$

Proof. First for any $\varphi \in C^3(\Delta)$, $\|\varphi\|_{C^3} \leq 1$, recalling that u^δ_τ is a probability measure, we clearly have

$$\int_\Delta \psi(p)\, du^\delta_\tau(p) \leq \int_\Delta |\psi(p)|\, du^\delta_\tau(p) \leq \int_\Delta du^\delta_\tau(p) = 1.$$

We deduce (6.17) taking the supremum over all such φ.

To prove (6.18) we write using (6.8) that for any $\varphi \in C^3(\Delta)$, $\|\varphi\|_{C^3} \leq 1$,

$$\begin{aligned}
&\int_\Delta \varphi(p)\, du^\delta_\tau(p) - \int_\Delta \varphi(p)\, du^\delta_{\tau'}(p) \\
&= \int_{\tau'}^{\tau} \int_\Delta \nabla\varphi(p)\mathcal{F}[u^\delta_s](p) du^\delta_s(p) ds + \frac{r^2}{2\delta} \int_{\tau'}^{\tau} \int_\Delta \sum_{i,j=1}^{d} Q_{ij}\partial_{ij}\varphi(p) G(p)^2\, du^\delta_s(p) ds \\
&+ \int_{\tau'}^{\tau} Error(s,\delta) ds.
\end{aligned} \tag{6.19}$$

Since Δ is bounded we have $|\mathcal{F}[u^\delta_s](p)| \leq C$. Thus

$$\begin{aligned}
&\Big| \int_\Delta \varphi(p)\, du^\delta_\tau(p) - \int_\Delta \varphi(p)\, du^\delta_{\tau'}(p) \Big| \\
&\leq C\Big(\|\nabla\varphi\|_\infty + r^2/\delta \|D^2\varphi\|_\infty + \frac{1}{\delta}\|D^3\varphi\|_\infty(\delta^3 + r^3)\Big)|\tau - \tau'| \\
&\leq C(1 + r^2/\delta).
\end{aligned}$$

Since we assumed that $r^2/\delta \to \lambda$, we obtain (6.18). □

We fix $T > 0$. The space $\mathcal{P}(\Delta)$ is compact for the weak convergence (and so for the norm $\|.\|_{sup}$). In view of the previous step we can apply Arzela-Ascoli Theorem to the sequence of continuous functions $u^\delta : [0, T] \to \mathcal{P}(\Delta)$ to obtain that a subsequence converges uniformly as $\delta \to 0$.

A diagonal argument shows in fact that a subsequence, that we still denote (u^δ), converges in $C([0,T],\mathcal{P}(\Delta))$ for any $T>0$ to some $v\in C([0,+\infty),\mathcal{P}(\Delta))$.

We now verify that v satisfies (4.5).

Step 3. v satisfies (4.5).

Proof. We need to pass to the limit in (6.8) as $\delta\to 0$. We fix some $\varphi\in C^3(\Delta)$ and $t>0$, and recall that $r^2/\delta\to\lambda$. Then it is easily seen the last term in the right hand side of (6.8) can be bounded by $C\|\varphi\|_{C^3}t(\delta+r.r^2/\delta)\to 0$. Let us pass to the limit in the second term in the right hand side. For a fixed $s\in[0,t]$, we write

$$\begin{aligned}\int_\Delta &\nabla\varphi(p)\mathcal{F}[u^\delta(s)](p)du^\delta_s(p)ds\\ &=\int_\Delta \nabla\varphi(p)(\mathcal{F}[u^\delta_s](p)-\mathcal{F}[v_s](p))du^\delta_s(p)ds\\ &\quad+\int_\Delta \nabla\varphi(p)\mathcal{F}[v_s](p)du^\delta_s(p)ds.\end{aligned}\tag{6.20}$$

Notice that

$$\mathcal{F}[u^\delta_s](p)-\mathcal{F}[u_s](p)=h(p)(p_ie_i^TAm^\delta(s)+m_i^\delta(s)e_i^TAp)$$

where

$$m^\delta(s)=\int_\Delta \tilde p\,d(u^\delta_s-u_s)(\tilde p).$$

Since $u^\delta_s\to v_s$ weakly uniformly in $s\in[0,t]$, we also have that

$$W_1(u^\delta_s,v_s)\to 0$$

uniformly in $s\in[0,t]$. Thus $m^\delta(s)\to 0$ and then $\mathcal{F}[u^\delta_s](p)\to\mathcal{F}[u_s](p)$ uniformly in $p\in\Delta$. We deduce that the first term in the right hand side of (6.20) goes to 0. The second term also goes to 0 since $\nabla\varphi$ and $\mathcal{F}[u_s]$ are Lipschitz.

Moreover,

$$\Big|\int_\Delta \nabla\varphi(p)\mathcal{F}[u^\delta_s](p)du^\delta_s(p)\Big|\le\|\nabla\varphi\|_\infty\|\mathcal{F}[u^\delta_s]\|_\infty\le C.$$

We then conclude that

$$\int_0^t\int_\Delta \nabla\varphi(p)\mathcal{F}[u^\delta_s](p)du^\delta_s(p)ds\to\int_0^t\int_\Delta \nabla\varphi(p)\mathcal{F}[u_s](p)du_s(p)ds$$

applying the Dominated Convergence Theorem. We can prove in the same way that the second term in the right hand side of (6.8) converges to

$$\lambda\int_0^t\int_\Delta\sum_{i,j=1}^d Q_{ij}\partial_{ij}\varphi(p)G(p)^2\,du_s(p)ds.$$

□

To conclude the proof it remains to study the case where $r^2/\delta^\alpha\to\lambda>0$ for some $\alpha\in(0,1)$. In that case we rescale time considering $\tau=\delta^\alpha t$. Reasoning as before we can write

$$\begin{aligned}\int_\Delta &\varphi(p)\,du^\delta_\tau(p)-\int_\Delta\varphi(p)\,du^\delta_{\tau'}(p)\\ &=\int_{\tau'}^\tau\frac{d}{d\tau}\int_\Delta\varphi(p)\,du^\delta_s(p)ds\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\delta^\alpha}\int_{\tau'}^{\tau}\int \mathbb{E}[\varphi(p^*)-\varphi(p)]du_s^\delta(p)du_s^\delta(\tilde p)\\
&= \delta^{1-\alpha}\int_{\tau'}^{\tau}\int_\Delta \nabla\varphi\mathcal{F}[u_s^\delta]\,du_s^\delta(p)+\frac{r^2}{2\delta^\alpha}\int_\Delta\sum_{i,j=1}^d Q_{ij}\partial_{ij}\varphi(p)G(p)^2\,du_s^\delta(p)\\
&\quad+\frac{1}{\delta^\alpha}\int_{\Delta^2}\mathbb{E}[R(p^*,p)]du_s^\delta(p)du_s^\delta(\tilde p).
\end{aligned}
$$

In view of (6.14) and recalling that we assume $r^2/\delta^\alpha\to\lambda$, the last term can be bounded by

$$C\|\varphi\|_{C^3}(\delta^{3-\alpha}+r.r^2/\delta^\alpha)+C\delta^{2-\alpha}\|D^2\varphi\|_\infty\le C\|\varphi\|_{C^3}(\delta+r).$$

It follows that step 6.2 still holds and thus, applying Arzela-Ascoli Theorem, we obtain that a subsequence, that we still denote (u^δ), converges in $C([0,T],\mathcal{P}(\Delta))$ for any $T>0$ to some $v\in C([0,+\infty),\mathcal{P}(\Delta))$. Passing to the limit $\delta\to0$ as in step 6.2, we deduce that v satisfies

$$\int_\Delta\varphi(p)\,dv_\tau(p)=\int_\Delta\varphi(p)\,dv_{\tau'}(p)+\int_{\tau'}^{\tau}\frac{\lambda}{2}\int_\Delta\sum_{i,j=1}^d Q_{ij}\partial_{ij}\varphi(p)G(p)^2\,dv_s(p)ds$$

which is the weak formulation of

$$\frac{d}{d\tau}v=\frac{\lambda}{2}\sum_{i,j=1}^d Q_{ij}\partial_{ij}(G^2\,v).$$

This concludes the proof of Theorem 4.2.

6.3. **Well-posedness of equation (4.6). Proof of Theorem 4.3.** Let us fix some $v\in C([0,+\infty],\mathcal{P}(\Delta))$ and denote $\mathcal{F}(t,p):=\mathcal{F}[v_t](p)$, namely

$$\mathcal{F}_i(t,p)=h(p)(p_ie_i^TA\bar p(t)+\bar p_i(t)e_i^TAp),\qquad \bar p(t)=\int_\Delta p\,dv_t(p).$$

Notice that $\vec{1}:=(1,\dots,1)\in\mathbb{R}^d$ is normal to Δ and that for any $p\in\Delta$,

$$\vec{1}\cdot\mathcal{F}(t,p)=\sum_{i=1}^d\mathcal{F}_i(t,p)=h(p)(p\cdot A\bar p(t)+\bar p(t)\cdot Ap)=0$$

since A is antisymmetric. Notice also that

$$\mathcal{F}(t,p)=0\qquad\text{for any } p\in\partial\Delta$$

since $h(p)=0$ if $p\in\partial\Delta$. It follows that any integral curve of $\mathcal{F}(t,p)$ starting from a point in Δ stays in the compact Δ forever. Let us denote $T^v_{s,t}$ the flow of the vector field $\mathcal{F}(t,p)$ namely

$$\frac{d}{dt}T^v_{s,t}(p)=\mathcal{F}[v_t](T^v_{s,t}(p)),\qquad T^v_{s,s}(p)=p.$$

We also let $T^v_t(p):=T^v_{0,t}(p)$. We thus just proved that

$$T^v_t(p)\in\Delta$$

for any $p\in\Delta$ and any $t\ge0$.

The characteristic method then allows to show in a standard way that the equation

$$\frac{d}{dt}u+\operatorname{div}(\mathcal{F}[v_t](p)u)=0$$

with initial condition $u_0 \in \mathcal{P}(\Delta)$ has a unique weak solution u in $C([0,+\infty],\mathcal{P}(\Delta))$ given by $u_t = \mathcal{T}_t^v \sharp u_0$ (see e.g. [46]).

Thus proving the existence of a unique weak solution to

$$\frac{d}{dt}v + \mathrm{div}(\mathcal{F}[v_t](p)v) = 0 \tag{6.21}$$

for a given initial condition $v_0 \in \mathcal{P}(\Delta)$ is equivalent to proving the existence of a solution to the fixed-point equation

$$v_t = T_t^v \sharp v_0. \tag{6.22}$$

This can be done applying the Banach fixed-point theorem to the map $\Gamma(v)_t := T_t^v \sharp v_0$ in the complete metric space $\{v \in C([0,T],\mathcal{P}(\Delta)),\, v(0) = v_0\}$ for a small enough $T > 0$ depending on $\|v_0\|_{TV}$. The details of the proof are an adaptation of e.g. [11]. The adaptation is almost straightforward noticing that $\mathcal{F}[v_t](p)$ is bounded Lipschitz in p uniformly in t for a given $v \in C([0,T],\mathcal{P}(\Delta))$. We can then repeat the process on $[T,2T]$, $[2T,3T]$,... to obtain $v \in C([0,+\infty],\mathcal{P}(\Delta))$ satisfying (6.22) which is thus the unique solution to (6.21) with initial condition v_0. The continuity with respect to initial conditions (4.7) can also be obtained slightly adapting the proof of [11]. We also refer to the recent paper [1] where well-posedness results for transport equations in measure spaces are obtained under general assumptions on the vector field.

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